Magnetohydrodynamics (MHD)

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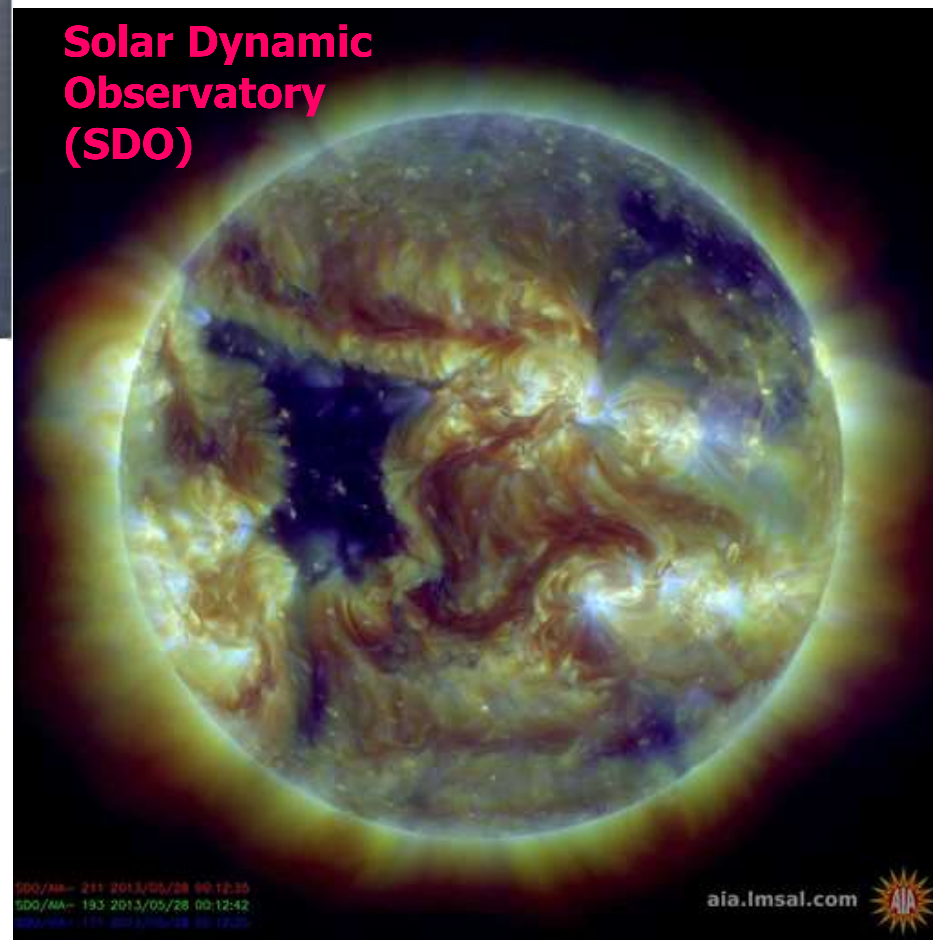
MagnetoHydroDynamics (MHD)

1. The MHD equations
2. Magnetic Reynolds number and ideal MHD
3. Some conservation laws
4. Static plasmas - magnetostatic equilibria and force-free fields
5. Flowing plasmas – the solar wind

The solar corona

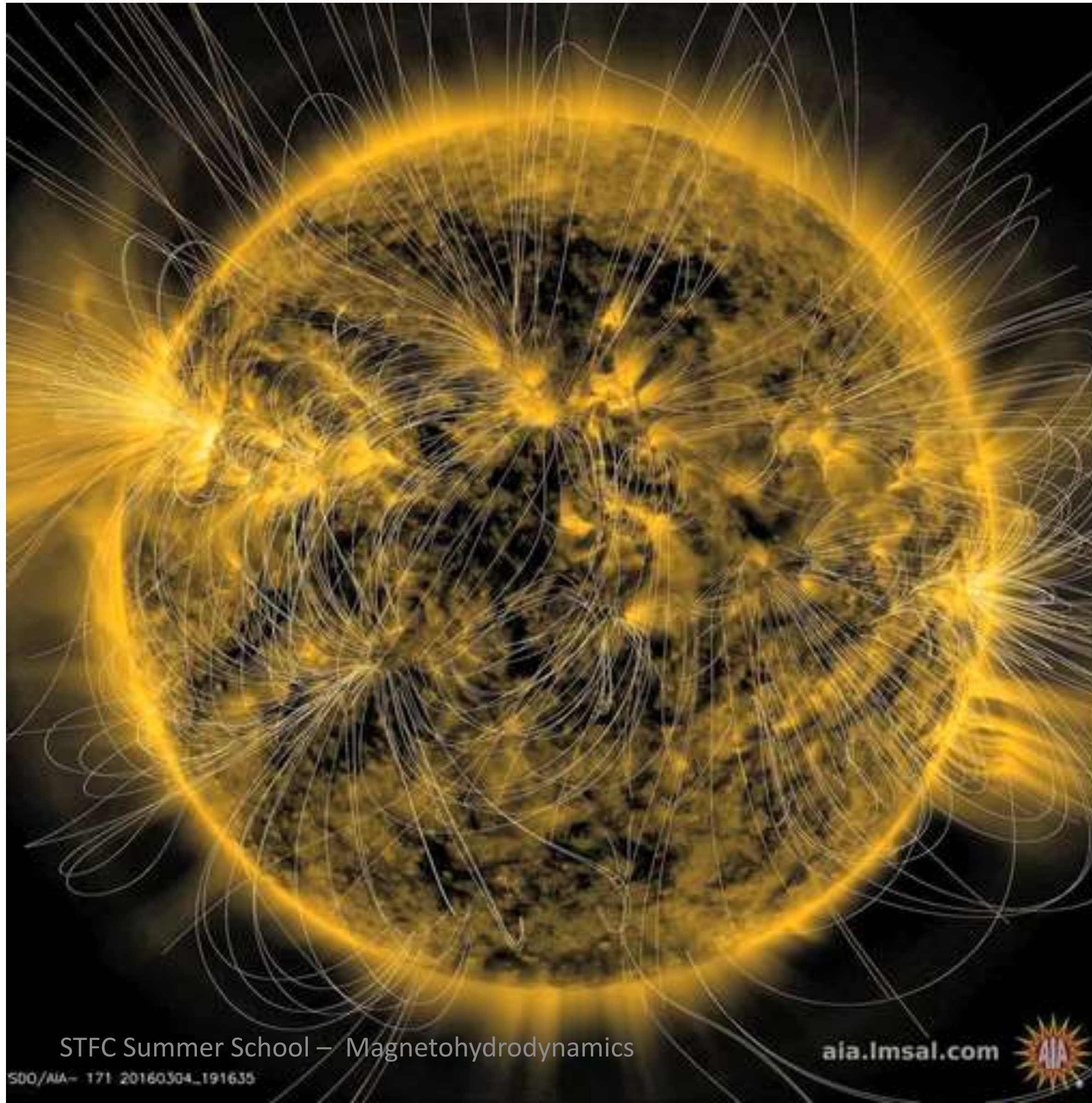
- The corona is the **hot** ($T \approx 10^6$ - 10^7 K) **tenuous** outer atmosphere of the Sun



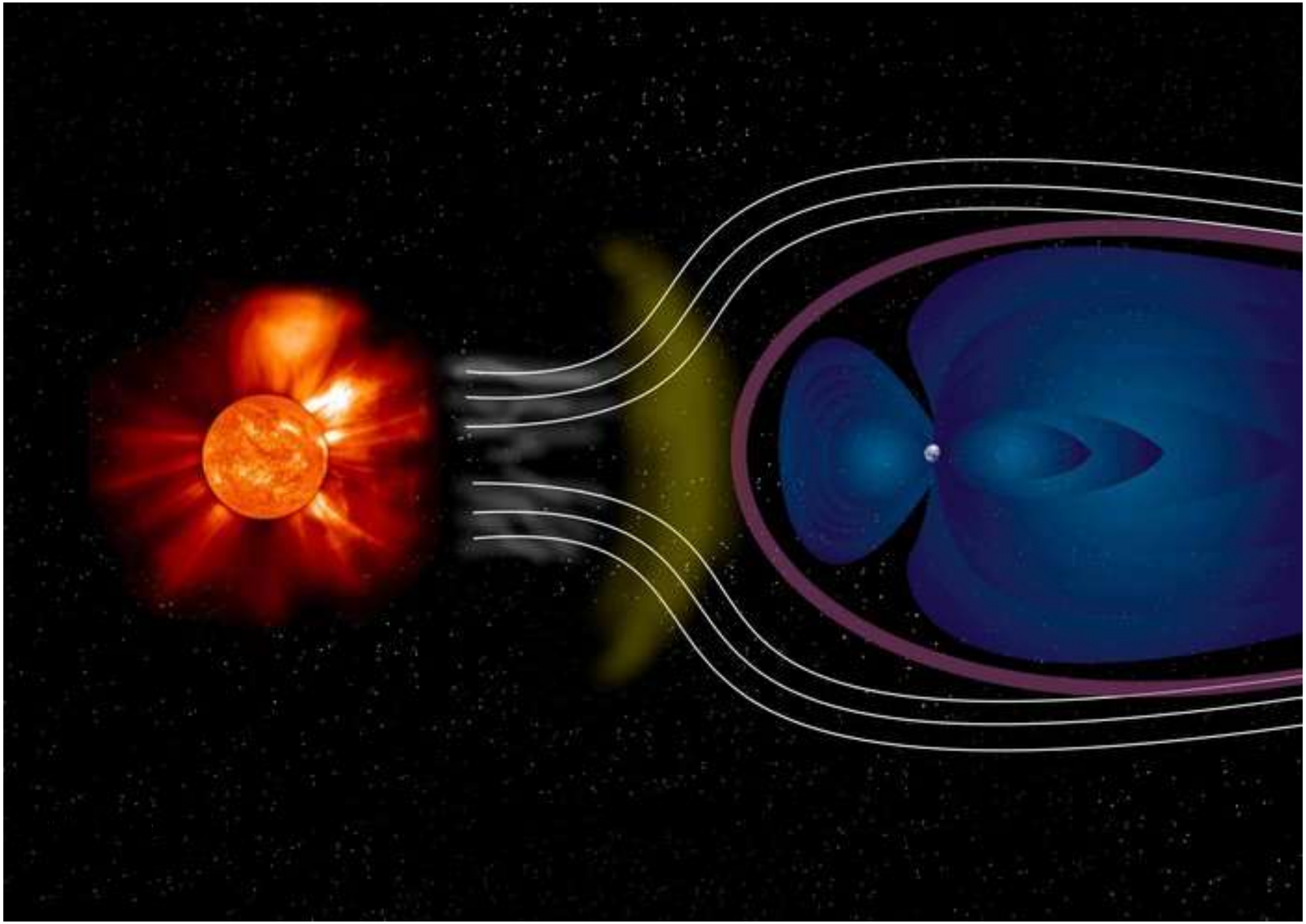
- Highly dynamic and structured
- Streaming into space as the solar wind

Magnetic fields in the solar corona

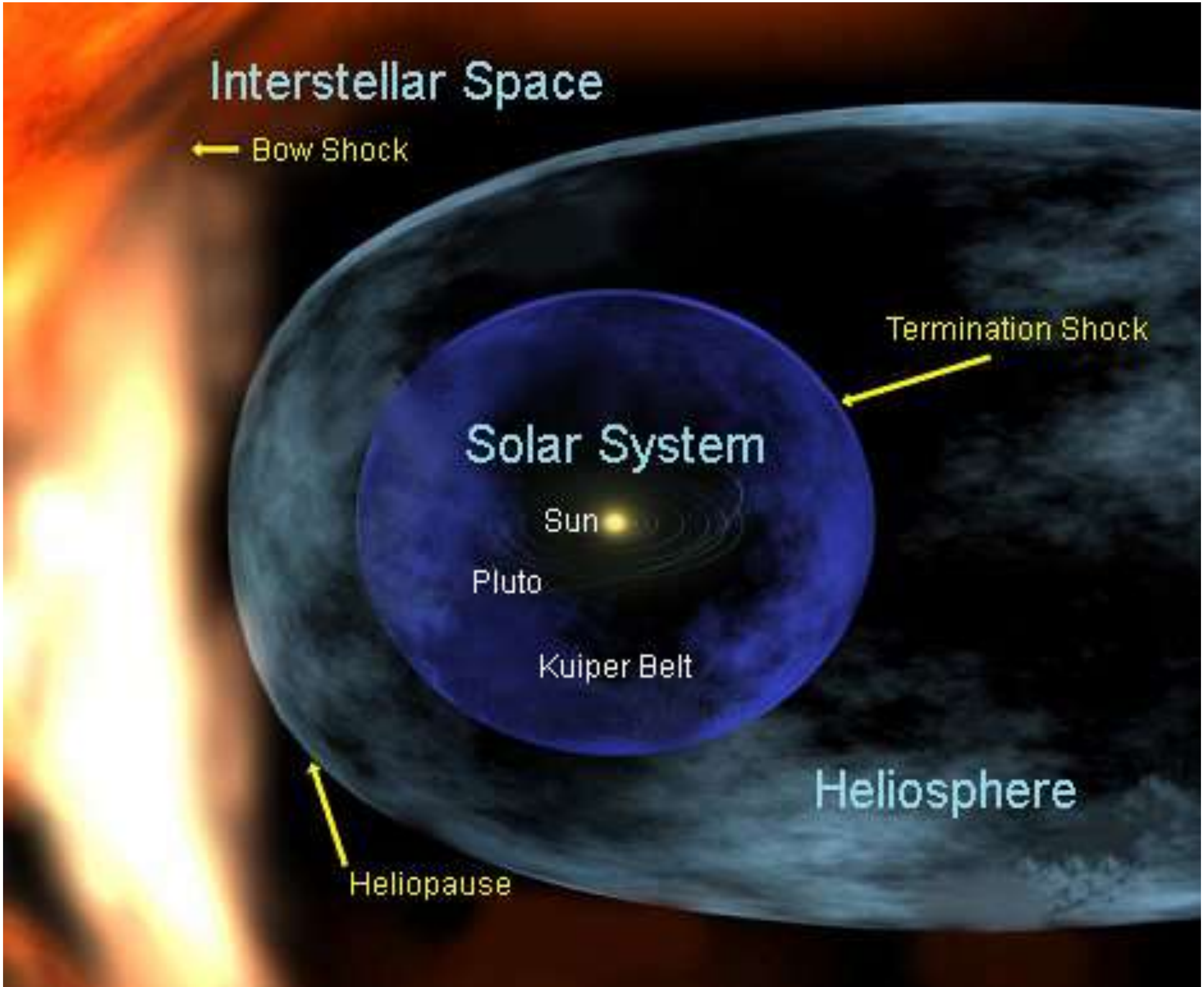
All structure and activity in solar atmosphere is controlled by magnetic field



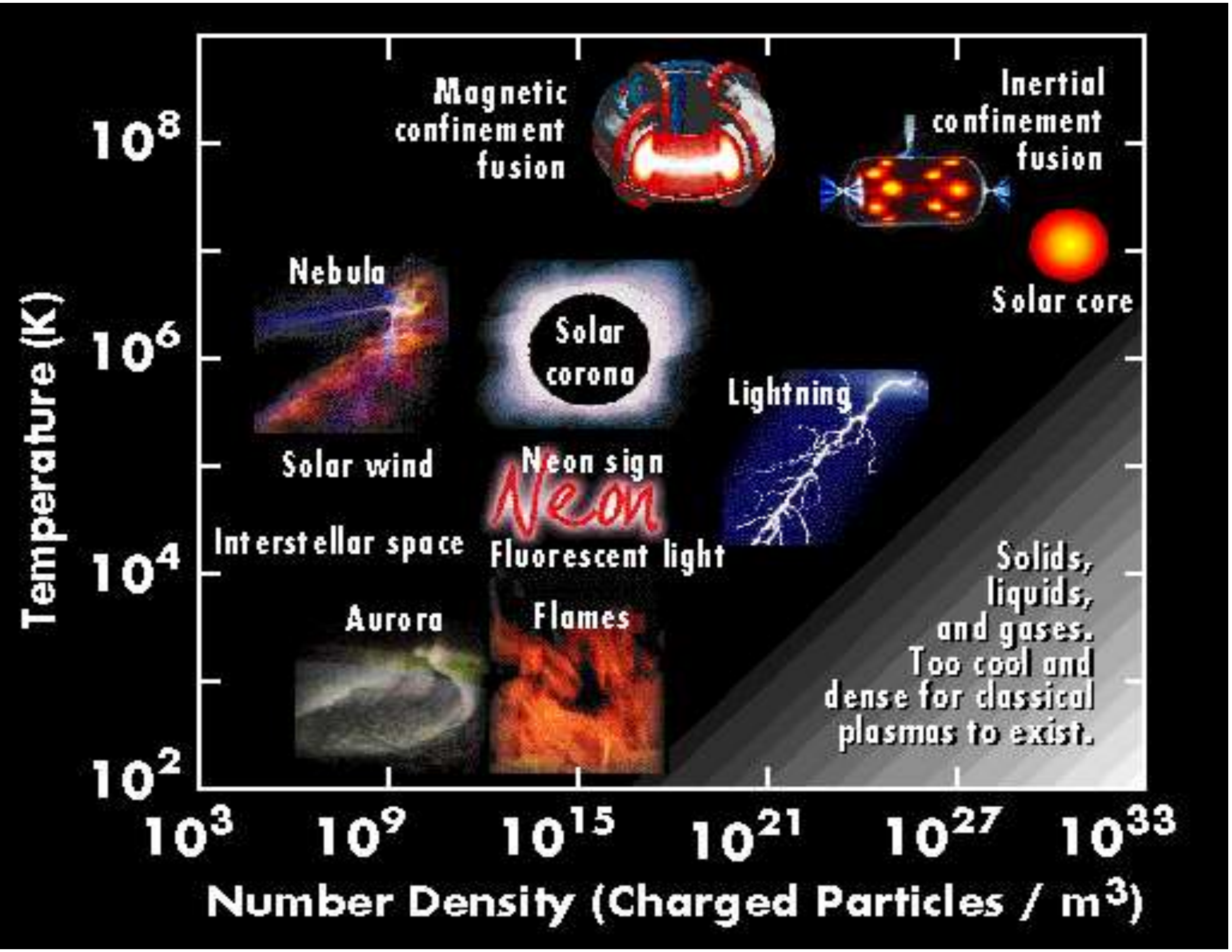
Solar wind interaction with Earth



The bigger picture – the Heliosphere



Plasma in the universe



- Plasma - a quasi-neutral gas of charged particles exhibiting collective behaviour
- > 99% of baryonic matter in universe is in plasma state
- Plasmas create magnetic fields and interact with magnetic fields

1. The magnetohydrodynamic (MHD) equations

The MHD model and its applicability

- One approach to modelling plasmas is **kinetic theory** – this models the distribution functions $f_s(\mathbf{r},\mathbf{v},t)$ for each species (s = "ions" or "electrons")

See Tsiklauri lecture

- From kinetic theory we may take moments (integrate over velocity space) and derive multi-fluid models (each species treated as a separate fluid) or single fluid models
- **MagnetoHydroDynamics** (**MHD**) treats the plasma as single electrically-conducting fluid which interacts with magnetic fields
- Does not consider separate behaviour of ions/electrons

Philosophy and validity of MHD

- Use equations of **fluid dynamics** and (pre) **Maxwell's equations**
- Plasma state specified by mass density ρ, temperature T, velocity **v** and magnetic field **B** at each point in space (**r**) and time t
- Mass density $\rho = nm \approx nm_i$; single temperature
  - For fully-ionised plasma; ignoring electron mass; charge neutrality $n_e = n_i \equiv n$
  - Local thermodynamic equilibrium; equal ion and electron temperatures

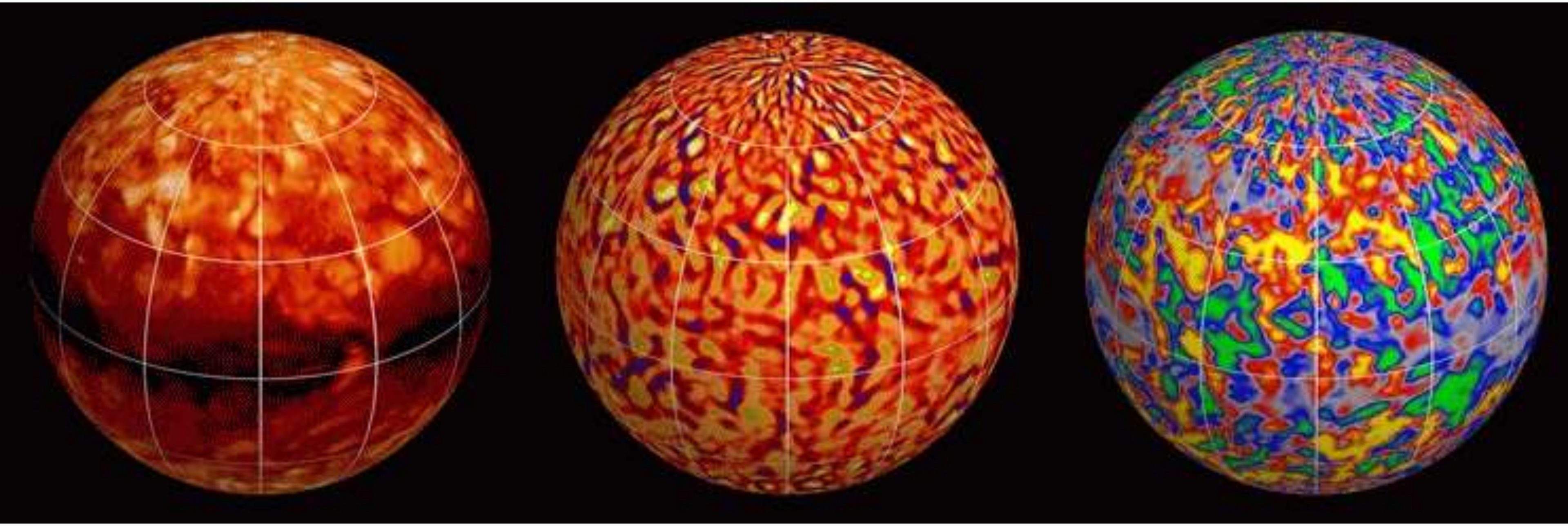
Collisional plasmas/**large-scale** phenomena ($r_L, \lambda_D, \lambda_{mfp} << L$)
Low frequency phenomena ($1/\tau << \omega_p, \omega_{ce}$)
Sub-relativistic ($u, L/\tau << c$) - neglect displacement current

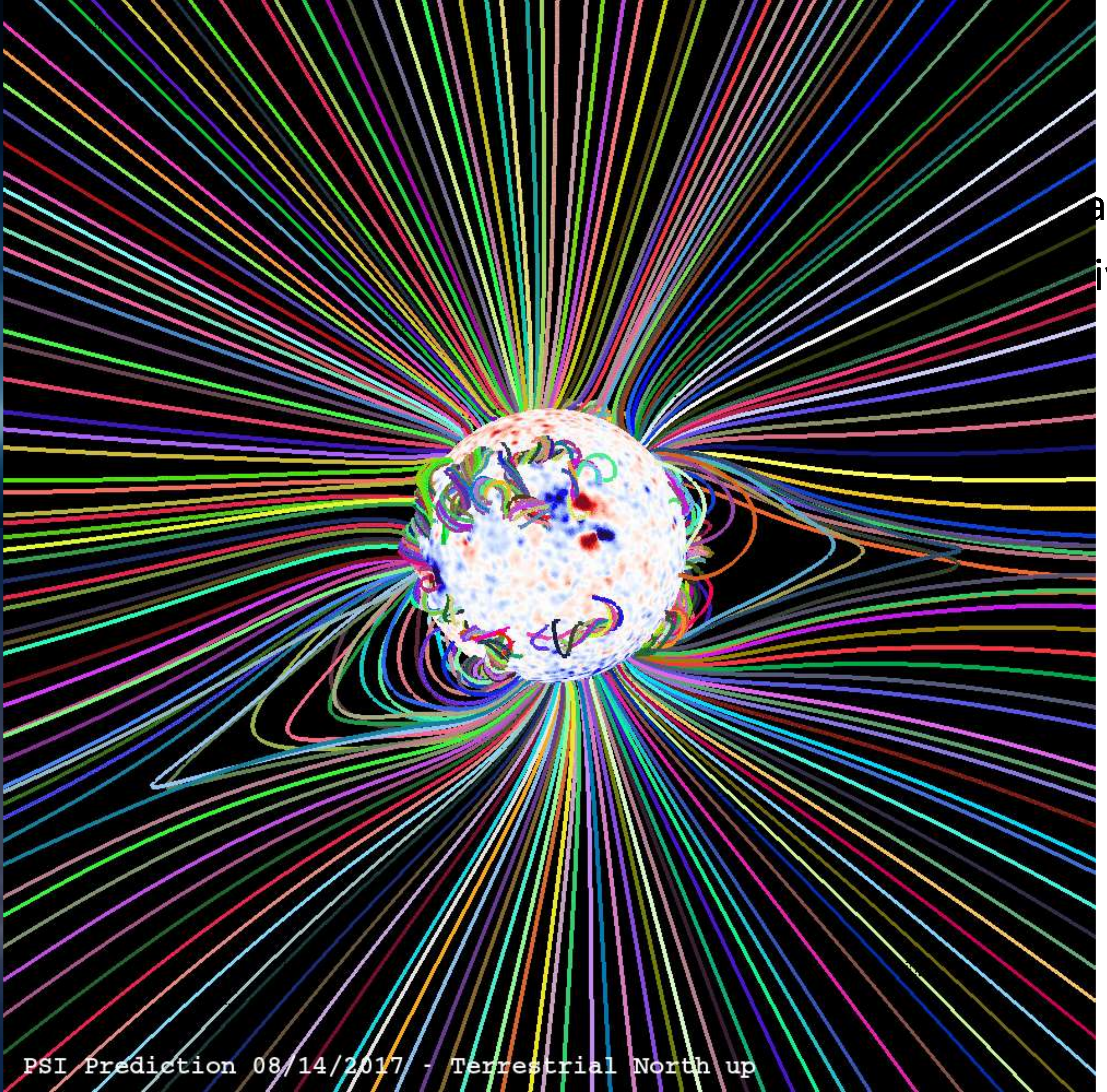
When is MHD useful?

- Widely used to model plasmas in magnetically-confined laboratory experiments, Sun, space, astrophysics
- Models interactions of plasma with magnetic fields, often in complex configurations – especially large-scale phenomena
- Also describes liquid metals
  - e.g. Earth's core, industrial processes
- Often gives useful information even when conditions for validity not strictly satisfied e.g. tokamaks, Earth's magnetosphere

3 m liquid metal dynamo experiment, U Maryland

MHD simulations of convection in solar interior





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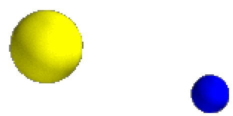
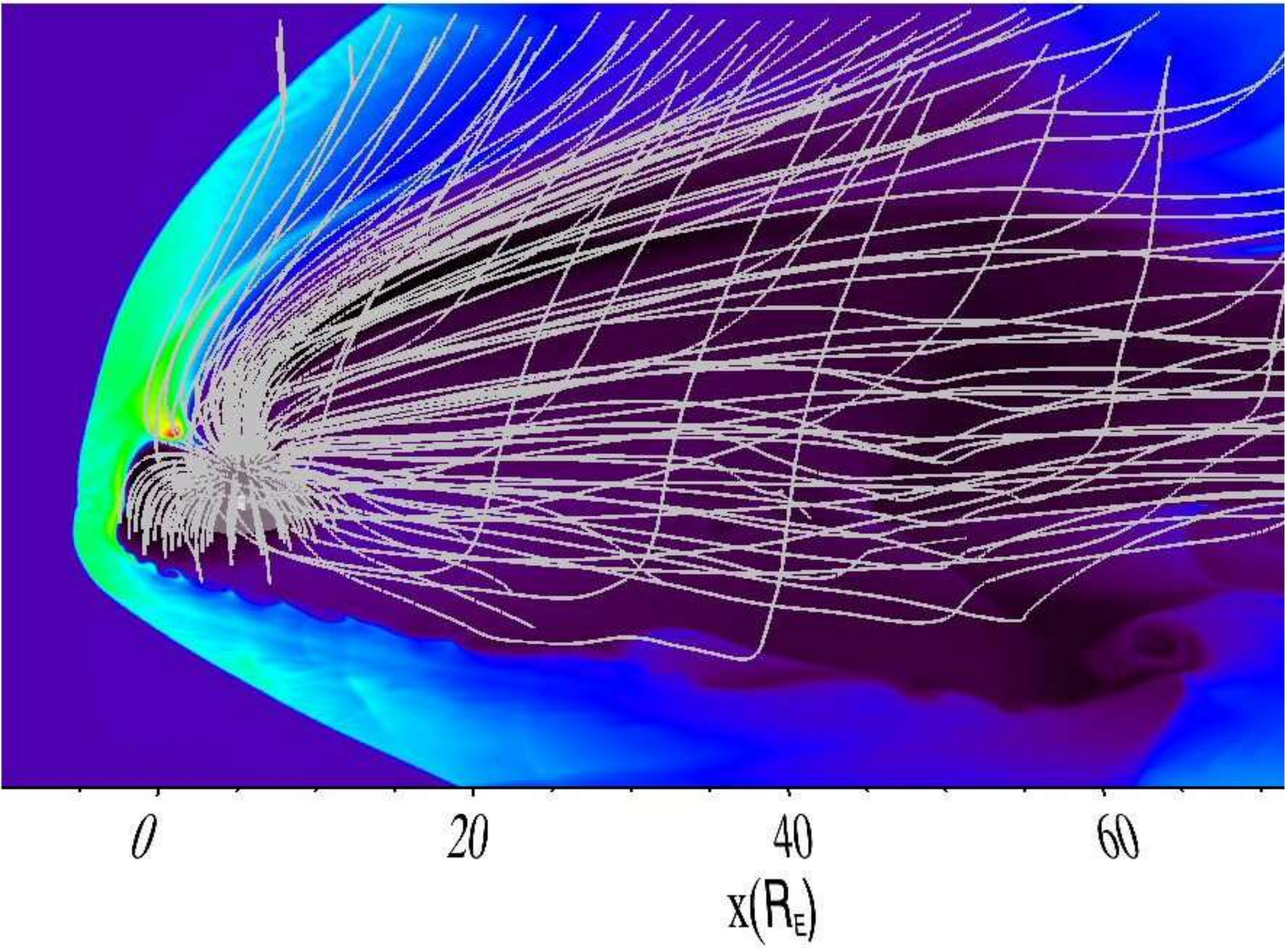
MHD simulation of solar wind



ENLIL simulation

http://www.spaceweather.eu/ccmc-enlil-help



Global MHD simulation of Earth's magnetosphere

Solar Terrestrial Environment Laboratory, Nagoya University

Derivation of MHD equations: Induction equation

Start from **Ampere's Law**

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j}$$

Neglecting displacement current – valid in non-relativistic limit *L*/*T* << *c*

Faraday's Law

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

and **Ohm's Law** (simple form)

$$\mathbf{j} = \sigma \left(\mathbf{E} + \mathbf{v} \times \mathbf{B} \right)$$

Electric field in reference frame of moving plasma

Conductivity

(neglecting electron inertia, Hall term etc)

Eliminating **j** and **E**, we can derive the **Induction Equation**:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}, \qquad \text{where } \eta = 1/(\mu_0 \sigma)$$

Magnetic diffusivity

- Determines evolution of magnetic field for given velocity field
- Note that conductivity σ is caused by collisions between ions and electrons and so is temperature-dependent. We use

$$\sigma = 7 \text{ X } 10^{-4} \ T^{3/2} \text{ mho m}^{-1} \text{ (or Ohm}^{-1} \text{ m}^{-1}) \text{ - where } T \text{ in K}$$

(but in derivation above we assume σ is constant)

Momentum equation

- The equation of motion or **momentum equation** is

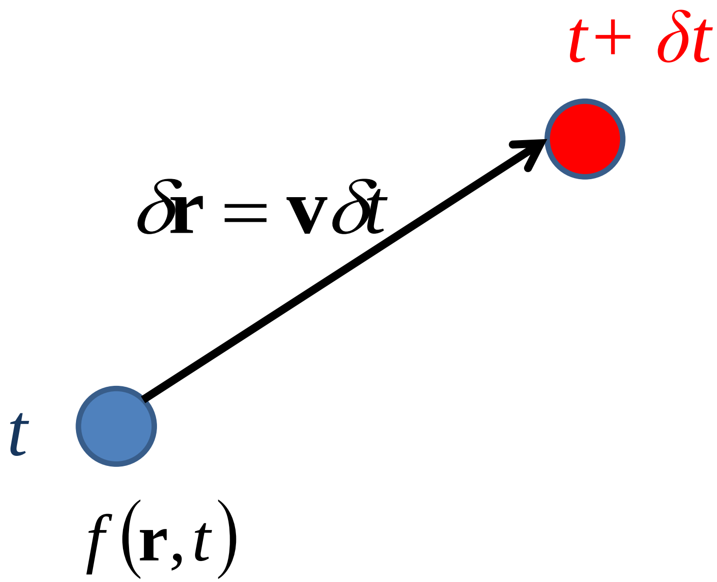
$$\rho \frac{d\mathbf{v}}{dt} = -\nabla p + \mathbf{j}\times\mathbf{B} + \rho\mathbf{g} + \mathbf{F}_{\text{visc}}$$

- The RHS is total force (per unit volume): gradient of pressure (p), Lorentz force, gravity and viscosity
  - Viscosity is anisotropic in magnetised plasma – but often negligible
  - Gravity negligible (usually) in laboratory plasmas
- Using Ampere's law, note

$$\mathbf{j}\times\mathbf{B} = \left(1/\mu_0\right)\left(\nabla\times\mathbf{B}\right)\times\mathbf{B}$$

- The LHS is the mass (per unit volume) multiplied by the acceleration seen by a moving fluid element – this is given by a convective derivative

$$\frac{d\mathbf{v}}{dt} = \left(\frac{\partial\mathbf{v}}{\partial t} + \mathbf{v}.\nabla\mathbf{v}\right)$$



$$f(\mathbf{r} + \delta\mathbf{r}, t + \delta t)$$

$$= f(\mathbf{r}, t) + \mathbf{v}\delta t.\nabla f + \delta t \frac{\partial f}{\partial t}$$

$$\Rightarrow \frac{df}{dt} = (\mathbf{v}.\nabla) f + \frac{\partial f}{\partial t}$$

Continuity equation and equation of state

- We have an equation of **mass conservation** or **continuity** (rate of change of mass in a small volume is equal to mass flux into volume – no sources/sinks of plasma):

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \mathbf{v}) = 0$$

- For **incompressible** flows this reduces to $\nabla \cdot \mathbf{v} = 0$
- The equation of state is usually taken as the **Perfect Gas Law**

$$p = (k_B / m)\rho T$$

where *m* is the mean particle mass.

- Note that for a fully-ionised hydrogen plasma (protons and electrons)

$$m \approx (1/2) m_p$$

Energy equation

- Finally we need an **energy equation**. Fairly generally this can be written

$$\frac{\rho^\gamma}{\gamma-1}(\frac{\partial}{\partial t}+\mathbf{v}.\nabla)(\frac{p}{\rho^\gamma}) = -\nabla.(\kappa.\nabla T) - \rho^2 Q(T) + \frac{j^2}{\sigma} + H$$

- LHS is rate of change of internal energy (per unit volume)
- RHS is sources/sinks of energy: **conduction** (where conductivity *κ* is related to collision-frequency and is also non-isotropic – different across field/along field); optically-thin **radiation**; **Ohmic** heating; other sources of heat (e.g. Viscous damping).
- Often we can use the simpler **adiabatic equation**

$$\frac{d}{dt}\left(\frac{p}{\rho^\gamma}\right) = 0$$

No magnetic monopoles

- In addition, Maxwell's equations give us a constraint on the magnetic field
- Magnetic monopoles do not exist:

$$\boxed{\nabla \cdot \mathbf{B} = 0}$$

The MHD equations

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}, \qquad (1.1)$$

$$\eta = \frac{1}{\mu_0 \sigma},$$

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}.\nabla \mathbf{v}\right) \equiv \rho \frac{d\mathbf{v}}{dt} = -\nabla p + \frac{1}{\mu_0}\left(\nabla \times \mathbf{B}\right) \times \mathbf{B}, \qquad (1.2)$$

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \mathbf{v}) = 0, \qquad (1.3)$$

$$p = (k_B / m)\rho T, \qquad (1.4)$$

$$\left(\frac{\partial}{\partial t} + \mathbf{v}.\nabla\right)\left(\frac{p}{\rho^\gamma}\right) = 0 \qquad (1.5)$$

$$\nabla \cdot \mathbf{B} = 0$$

The MHD equations

- 9 highly-nonlinear, coupled equations in 9 unknowns (components of **B**, **v**; p, ρ, T)
- Secondary variables – notably current and electric field **j** and **E** – can be derived as required (from Ohm's Law, Ampere's Law)

- Plasma flows → magnetic field (induction equation)
- Magnetic field → plasma flows (momentum equation)

- More general versions are available e.g.
  - Multi-fluid (Hall-MHD - treats two-fluid effects; add neutral fluid, etc)
  - Relativistic MHD (used in high-energy astrophysics)
  - Radiative MHD (couple with radiative transfer)

The momentum equation and Lorentz force

- Using the vector identity

$$\nabla(\mathbf{A}\cdot\mathbf{B})=(\mathbf{B}\cdot\nabla)\mathbf{A}+(\mathbf{A}\cdot\nabla)\mathbf{B}+\mathbf{A}\times(\nabla\times\mathbf{B})+\mathbf{B}\times(\nabla\times\mathbf{A})$$

and setting $\mathbf{A} = \mathbf{B}$ (using also $\mathbf{B.B} = B^2$) we obtain an expression for the Lorentz force:

$$\mathbf{j}\times\mathbf{B}=\frac{1}{\mu_0}(\nabla\times\mathbf{B})\times\mathbf{B}=-\nabla\left(\frac{B^2}{2\mu_0}\right)+(\mathbf{B}\cdot\nabla)\mathbf{B}$$

= magnetic pressure force + magnetic tension force

- **Magnetic pressure** is $B^2/2\mu_0$

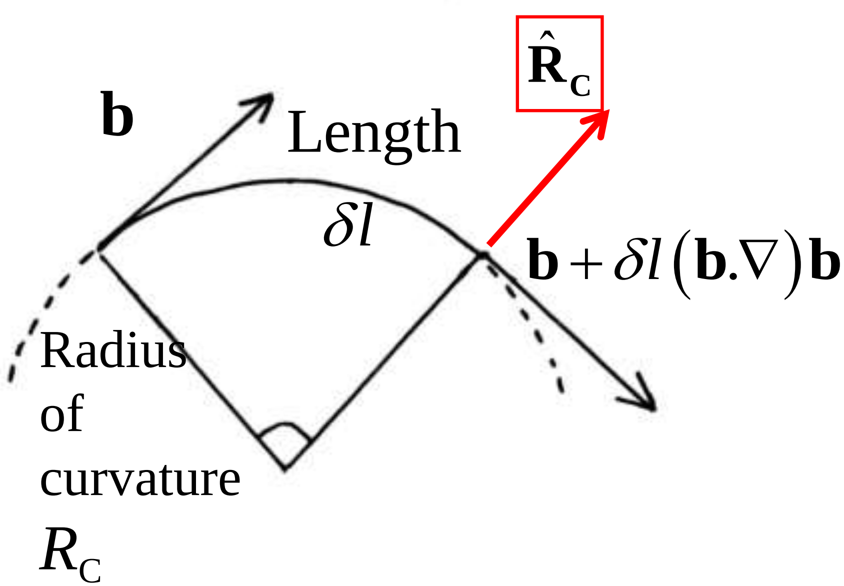
Total pressure = magnetic pressure + thermal pressure

= $B^2/2\mu_0 + p$

Magnetic tension

$$\frac{1}{\mu_0}(\mathbf{B}\cdot\nabla)\mathbf{B} = \frac{B^2}{\mu_0}(\mathbf{b}.\nabla)\mathbf{b} \qquad (\text{where } \mathbf{b} = \mathbf{B}/B)$$

- Magnetic tension force acts towards centre of curvature of field lines
- Magnetic field lines are like “stretched strings”



$$(\mathbf{B}.\nabla)\mathbf{B} = -\frac{B^2}{R_c}\hat{\mathbf{R}}_c$$

Similar to “centrifugal acceleration” associated with curved velocity streamlines $(\boldsymbol{v}.\boldsymbol{\nabla})\boldsymbol{v}$

Plasma beta

- The relative important of the pressure force and magnetic force is determined by the “plasma beta”

$$\beta = \frac{\text{Thermal pressure}}{\text{Magnetic pressure}} = \frac{p}{B^2/2\mu_0} = \frac{2\mu_0 p}{B^2} \qquad (1.6)$$

- In many instances (e.g. solar corona...) β is small – magnetic forces dominate

2. MAGNETIC REYNOLDS NUMBER AND IDEAL MHD

More about the induction equation

- What is the relative importance of plasma motions and Ohmic resistivity in induction equation?

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}$$

(1) (2)

- Consider ratio of terms (1) and (2) on RHS:

$$\frac{(1)}{(2)} \approx \frac{vB/L}{\eta B/L^2} = \frac{Lv}{\eta}$$

where *L* and *v* are typical length-scale and velocity

- This is the **magnetic Reynolds number** *(Re)* - a dimensionless number similar to Reynolds number in fluid dynamics

$$\mathrm{Re} = \frac{Lv}{\eta} = \mu_0 \sigma L v$$

Magnetic Reynolds number

- Magnetic Reynolds number tells us the relative importance of resistivity and plasma flows in determining evolution of **B**

Re >> 1 :   flows dominate, resistivity negligible

Re << 1:   highly-resistive

Fusion and astrophysical plasmas are very good conductors, also *L* very large in astrophysical plasmas

→ ***Re* usually very large**

- Consider two limits separately

The limit Re << 1 (high resistivity)

- Induction equation becomes diffusion equation

$$\frac{\partial \mathbf{B}}{\partial t} = \eta \nabla^2 \mathbf{B} \qquad \frac{B}{t_d} \approx \eta \frac{B}{L^2}$$

- Field gradients/currents diffuse away due to Ohmic resistivity on a **diffusion time-scale**

$$t_d = \frac{L^2}{\eta}$$

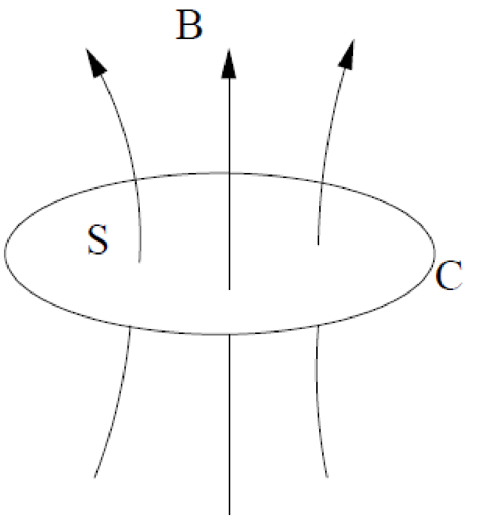
- Usually very slow process e.g. in solar corona, taking L = 1000 km, T = 1 million K

$$t_d \approx 30{,}000 \text{ years}$$

Ideal MHD : Re >> 1 (high conductivity)

- The limit Re >> 1 – (high conductivity) is known as **Ideal MHD**
- Relevant for tokamaks and astrophysics
  - Note this is a singular limit in some senses (see magnetic reconnection , later)
- The induction equation becomes

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B})$$



$$\Phi = \int_S \mathbf{B}.d\mathbf{S}$$

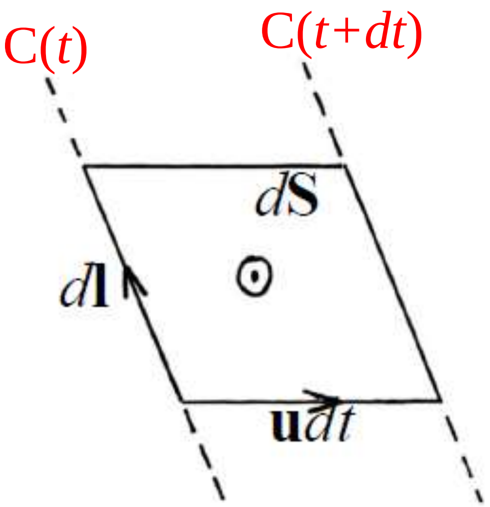
- Consider rate of change of magnetic flux through a **fluid surface** S, bounded by closed curve C

Flux Φ may change due to –
•Field $\mathbf{B}$ changing in time
•Surface S moving/distorting as fluid moves

$$\frac{d\Phi}{dt} = \int_S \frac{\partial \mathbf{B}}{\partial t} . d\mathbf{S} + \int_C \mathbf{B} . \left(\mathbf{v} \times d\mathbf{l} \right)$$

(1) Due to explicit local time variation of **B**

(2) Due to motion of surface S



Moving fluid sweeps out area **v**dt X d**l** in time dt

Substitute from induction equation (1) and re-arrange vector triple product (2) →

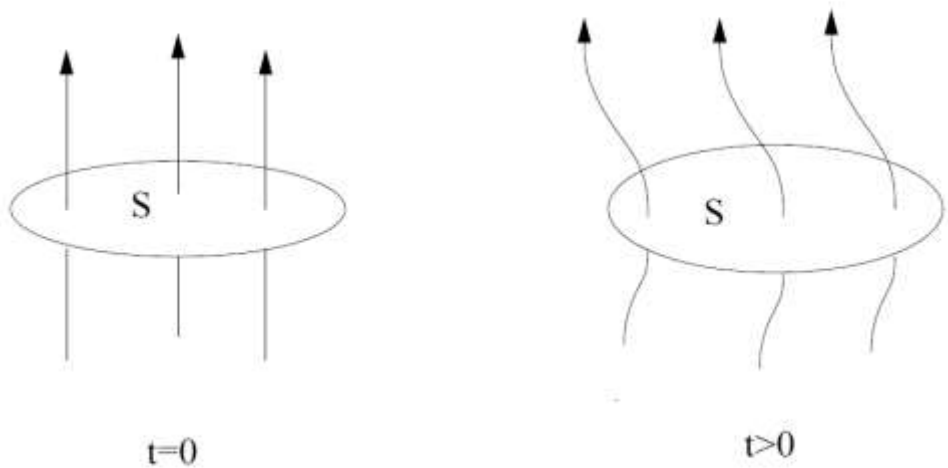
$$\frac{d\Phi}{dt} = \int_S \nabla \times \left(\mathbf{v} \times \mathbf{B} \right) . d\mathbf{S} + \int_C \left(\mathbf{B} \times \mathbf{v} \right) . d\mathbf{l}$$

Use Stokes' Theorem on term (2) →

$$\frac{d\Phi}{dt} = \int_S \nabla \times \left(\mathbf{v} \times \mathbf{B} \right) . d\mathbf{S} + \int_S \nabla \times \left(\mathbf{B} \times \mathbf{v} \right) . d\mathbf{S} = 0$$

Alfven's Theorem

- **In a perfectly-conducting plasma, the magnetic flux through any fluid surface is conserved**.



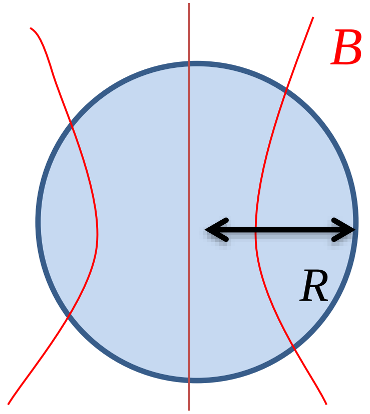
Example A star of initial radius 10^6 km and magnetic field strength 0.01 T collapses to form a neutron star of radius 10 km. Estimate the magnetic field in the neutron star.

$$\Phi \approx BA \text{ constant } \Rightarrow B_1 \pi R_1^2 = B_2 \pi R_2^2$$

Hence final magnetic field $B_2 = B_1 \left(R_1 / R_2 \right)^2$

$$= 0.01 \times \left(10^5\right)^2 = 10^8 \text{ T}$$

Very strong field! A pulsar.

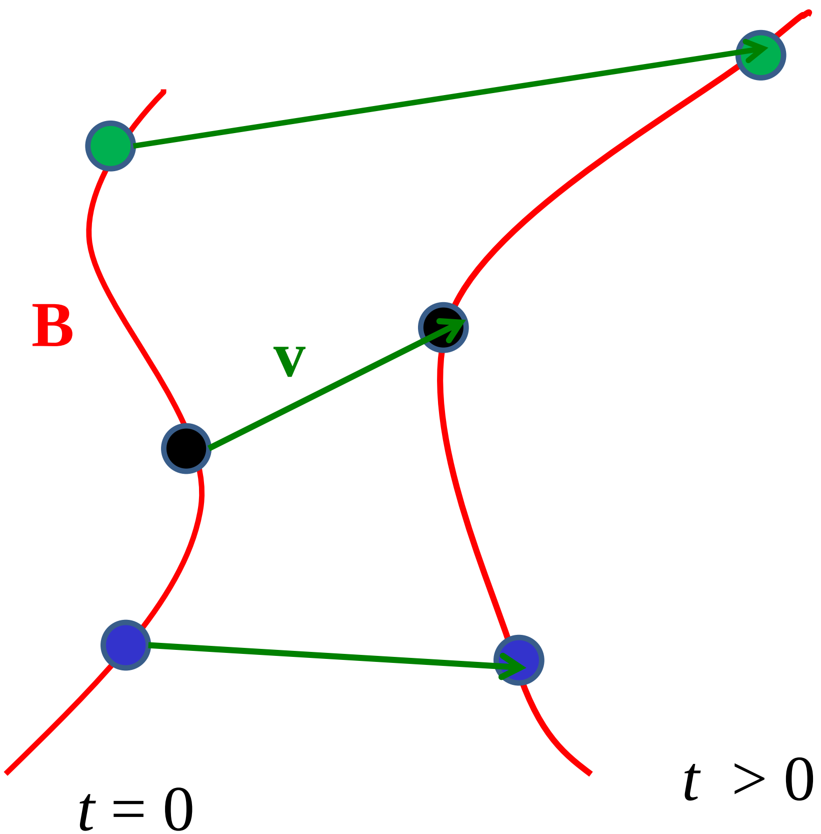


Frozen-in fields

- In ideal MHD, magnetic fieldlines move with the fluid:

Magnetic field is frozen to the plasma

- The topology/connectivity of the magnetic field is invariant

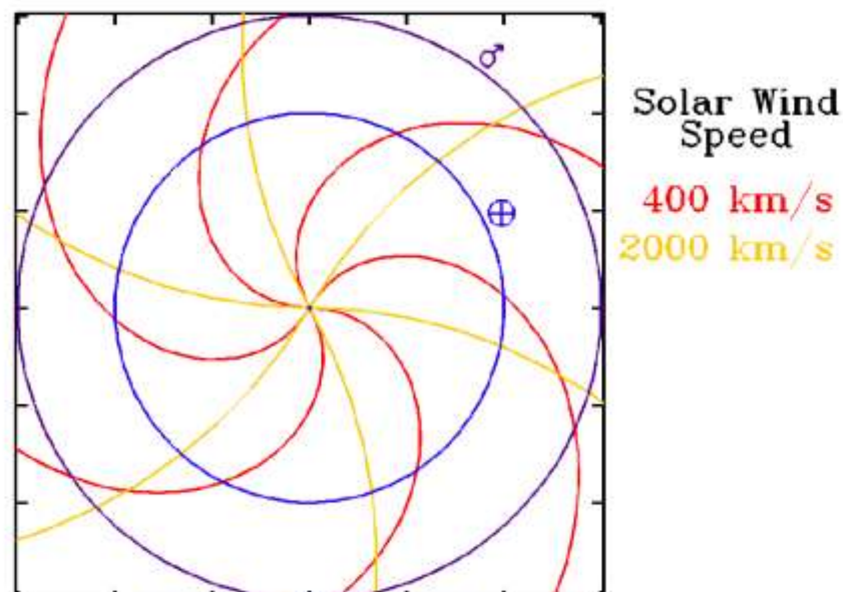


Example of frozen-in fields –
the Solar Wind magnetic field

- The solar wind is a supersonic flow of plasma away from the Sun - typical speed 350 km s^{-1} (*see Section 5*)
- The Sun also rotates every 27 days or so, giving an angular velocity of about 2.7 X 10^{-6} rad s^{-1}
- The magnetic field is frozen to the spinning outflowing plasma which forms spiral pattern (like garden sprinkler!)

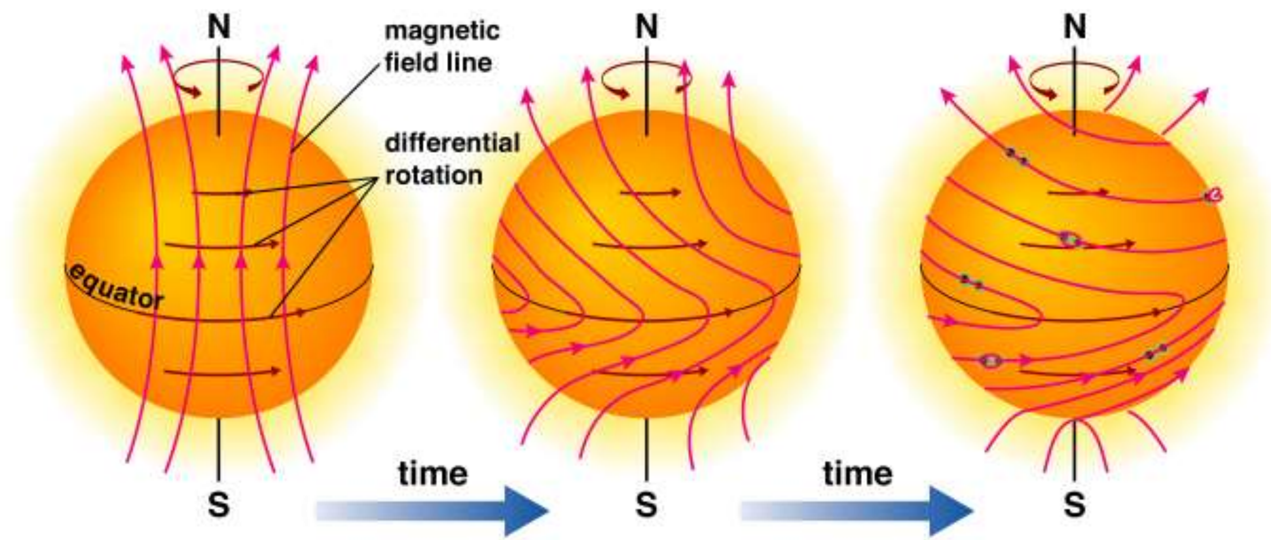
$$B_\theta / B_r = r\omega / v_{sw}$$

At 1 au this gives a spiral angle ψ (where $\tan \psi = B_\theta/B_r$ about 45°



Example of frozen-in fields -dynamos

- Magnetic field in Sun, stars, galaxies, planetary interiors usually generated by a **dynamo** – plasma motions create magnetic field

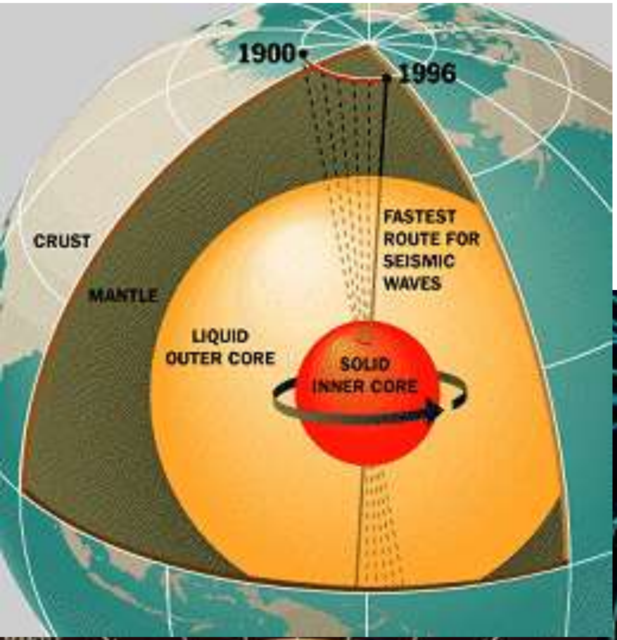


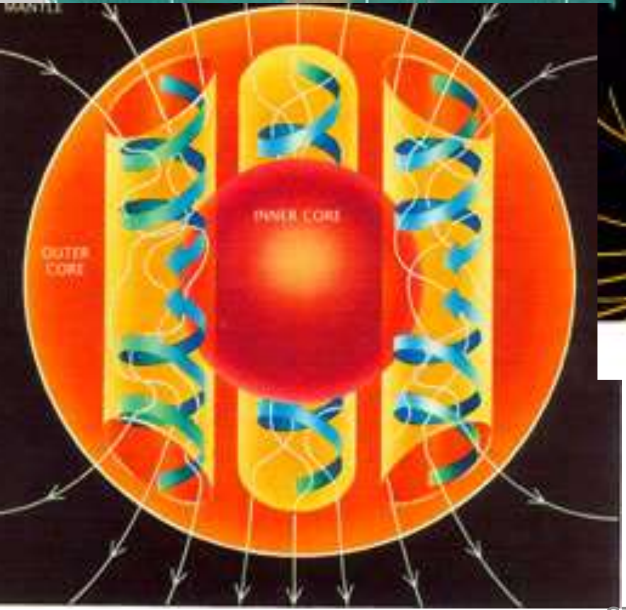
•Differential rotation generates toroidal field (EW) from poloidal field (NS)

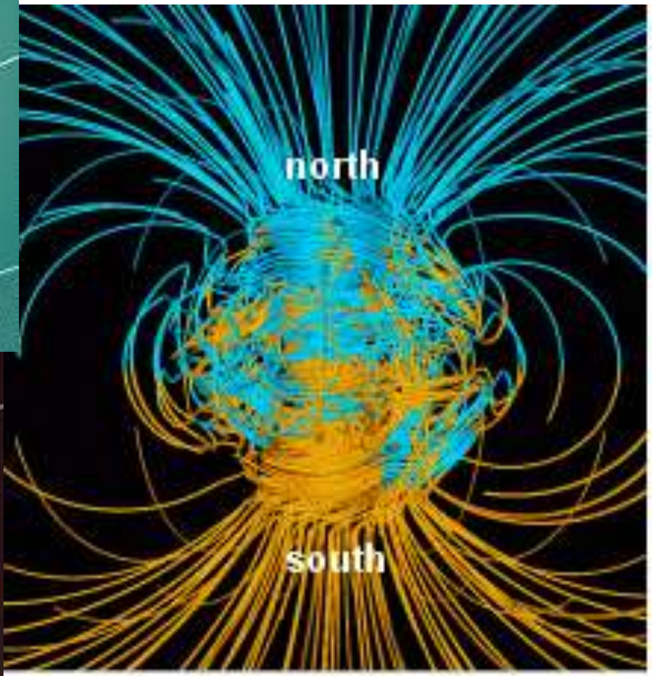
•Also need mechanism to generate poloidal field from toroidal e.g. Associated with turbulence

See e.g. P Charbonneau, Living review of Solar Physics (2010)

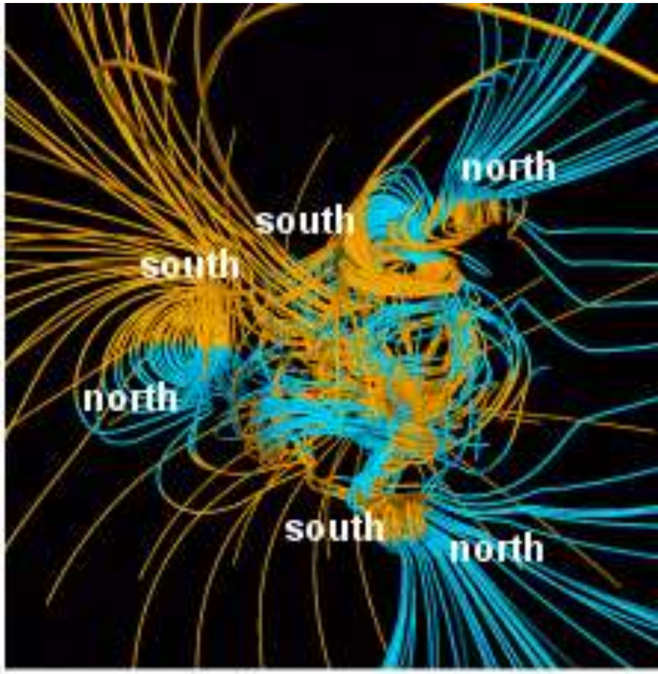
Geomagnetic dynamo







between reversals

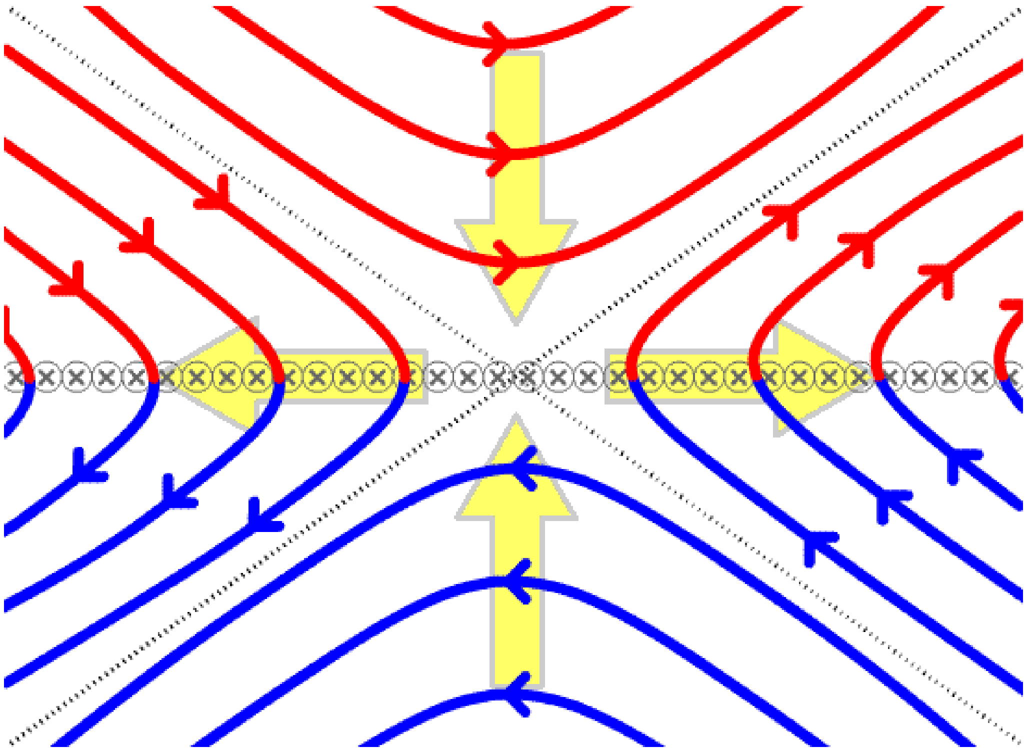


during a reversal

Magnetic reconnection

- Even in very highly-conducting plasmas, the frozen-in condition can be violated locally
- Magnetic field lines can break and reconnect at current sheets

- Ideal outer region
- Inner resistive region - a thin layer

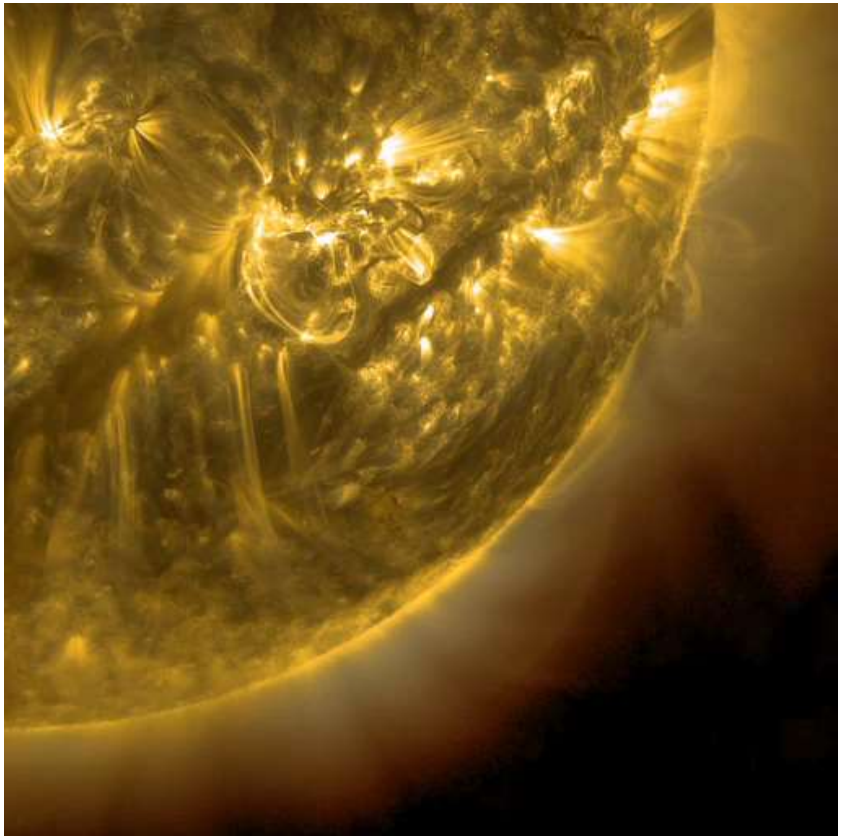


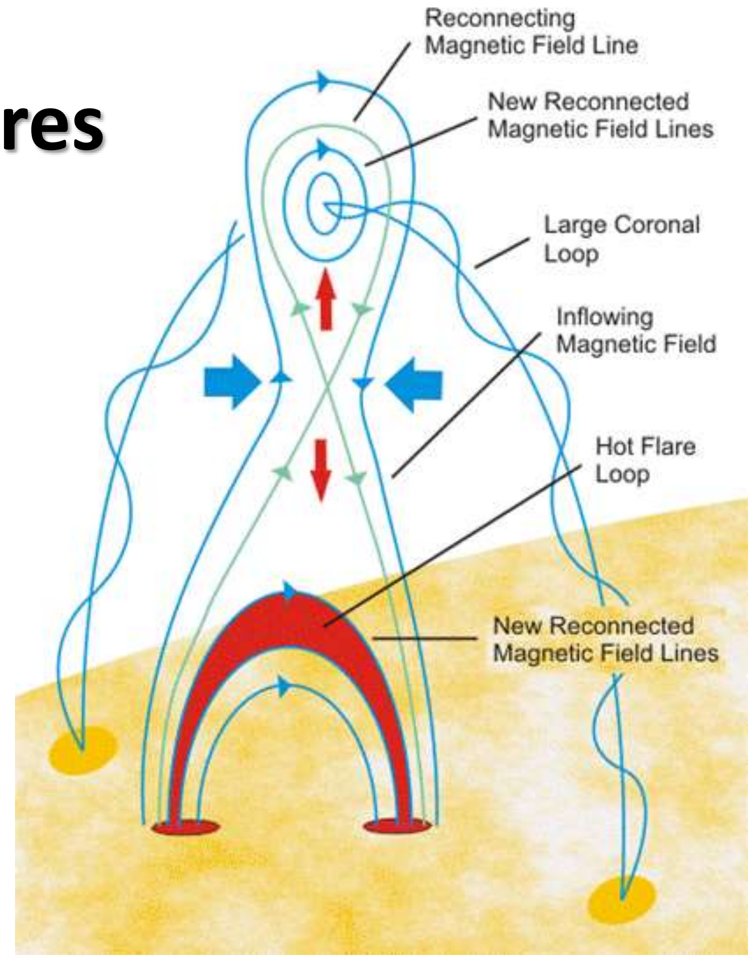
Important in solar flares, magnetospheres, tokamaks etc
Restructures magnetic field and efficiently dissipates magnetic energy

See Hornig lecture

Reconnection in action-Solar flares

X class flare Dec 2014





- Solar flares are dramatic events releasing up to 10^{25} J of stored magnetic energy over minutes/hours
- Flares generate plasma heating and fast particle beams - signatures across the em spectrum from gamma rays to radio - *see Kontar and Matthews lectures*

3. CONSERVATION LAWS

Energy

Conservative form of MHD equations

Helicity

Energy

- Consider flows of energy in **ideal MHD**

Dot product momentum equation with **v**

$$\rho \mathbf{v}.\left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}.\nabla)\mathbf{v}\right] = -(\mathbf{v}.\nabla)p + \frac{1}{\mu_0}\mathbf{v}.\left[(\nabla \times \mathbf{B}) \times \mathbf{B}\right]$$

Using vector identities and mass conservation

$$\Rightarrow \frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) + \nabla.\left(\frac{1}{2}\rho v^2 \mathbf{v}\right) = -(\mathbf{v}.\nabla)p + \frac{1}{\mu_0}\mathbf{v}.\left[(\nabla \times \mathbf{B}) \times \mathbf{B}\right]$$

Rate of change of kinetic energy

Flux of kinetic energy

Work done by pressure gradient and Lorentz force

$$\frac{\partial}{\partial t}\left(\frac{1}{2\mu_0}B^2\right) + \nabla.\left(\frac{\mathbf{E} \times \mathbf{B}}{\mu_0}\right) = 0$$

Magnetic energy balance from Ampere's and Faraday's Law – no Ohmic dissipation

Rate of change of magnetic energy

Poynting flux

Combining above equations with adiabatic energy equation (1.5) – and a little algebra - gives **ideal energy conservation law**

$$\Rightarrow \frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2 + \frac{1}{2\mu_0}B^2 + \frac{p}{\gamma - 1}\right) + \nabla . \left(\frac{1}{2}\rho v^2 \mathbf{v} + \frac{\mathbf{E}\times\mathbf{B}}{\mu_\mathbf{0}} + \frac{\gamma}{\gamma - 1} p\mathbf{v}\right) = 0 \quad (3.1)$$

Rate of change of total energy
Kinetic + magnetic +internal

Flux of K.E.

Poynting flux

Enthalpy flux

Integrating over volume and assuming surface integral (integrated energy flux) vanishes at infinity (boundary of region of interest), we obtain conservation law:

$$K + W = \text{constant}$$

Kinetic energy + potential energy = constant

$$K = \int_V \frac{1}{2}\rho v^2 dV, \qquad \mathrm{W} = \int_V \left(\frac{1}{2\mu_0}B^2 + \frac{p}{\gamma - 1}\right) dV, \quad (3.2)$$

Kinetic energy

Potential energy

Conservative form of MHD equations

- The ideal MHD equations may be written in conservative form

$$\frac{\partial}{\partial t}\left(\text{quantity}\right)+\nabla.\left(\text{flux}\right)=0$$

- Useful for simulations, also shock theory (see Gordovskyy lecture)

Mass (1.3):

$$\frac{\partial}{\partial t}\left(\rho\right)+\nabla.\left(\rho\mathbf{v}\right)=0$$

Momentum – rearranging (1.4):

$$\frac{\partial}{\partial t}\left(\rho\mathbf{v}\right)+\nabla.\left(\rho\mathbf{v}\mathbf{v}+\left(p+\frac{B^2}{2\mu_0}\right)\mathbf{I}-\frac{1}{\mu_0}\mathbf{B}\mathbf{B}\right)=0$$

Energy (3.1):

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2+\frac{p}{\gamma-1}+\frac{B^2}{2\mu_0}\right)+\nabla.\left(\frac{1}{2}\rho v^2\mathbf{v}+\frac{\gamma p}{\gamma-1}\mathbf{v}+\frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}\right)=0$$

Magnetic helicity

- Magnetic helicity is a useful global quantity which quantifies field topology

$$K = \int_V \mathbf{A}.\mathbf{B}\, dV \quad \text{where } \nabla \times \mathbf{A} = \mathbf{B} \qquad (3.3)$$

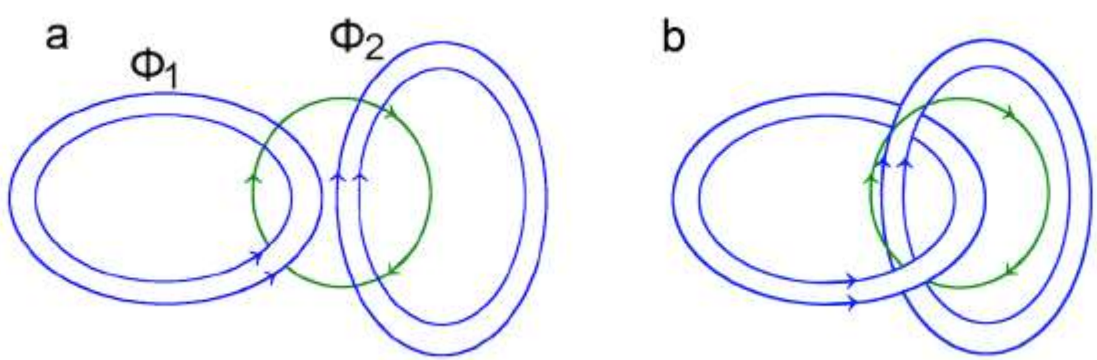
A is vector potential

- It can be shown that for flux tubes with magnetic fluxes Φ_1, Φ_2, interlinking N times

$$K = \pm N \Phi_1 \Phi_2$$

L Woltjer, Proc Nat Acad Sci (1958); K Moffatt "Magnetic field generation in electrically conducting fluids" (1978)

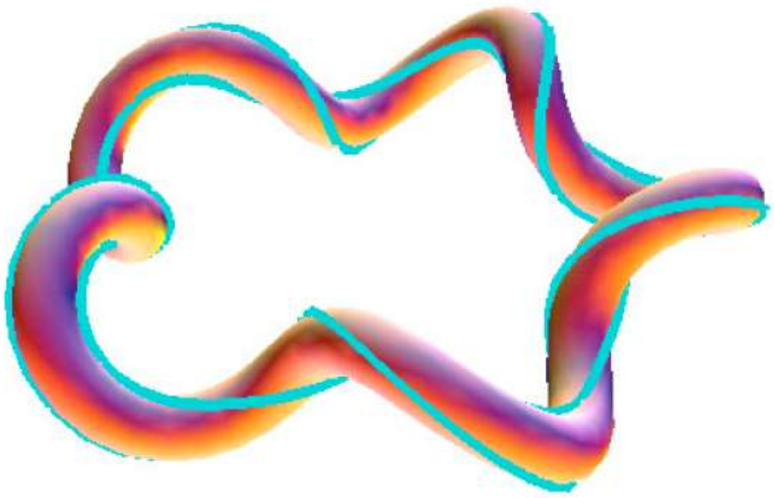
$$K = 0$$



$$K = \Phi_1 \Phi_2$$

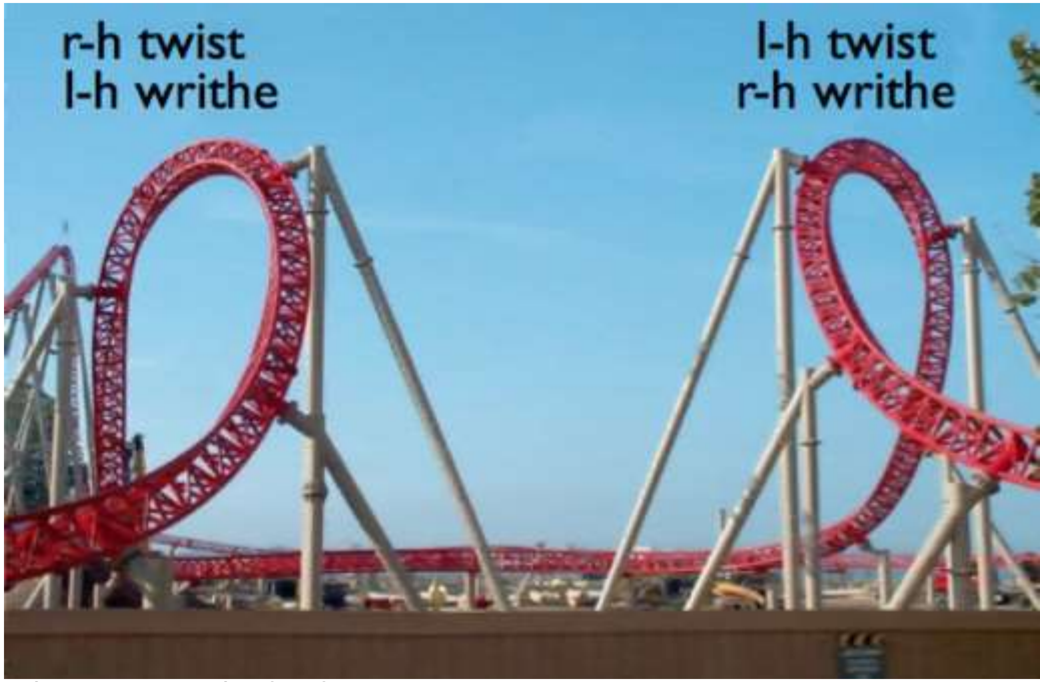
Equivalently a single flux rope with twist T (no. of poloidal turns = $1/q$) and "writhe" (axis distortion) W_r has helicity

$$K = (T + W_r)\Phi^2$$



A $m = 5$ distorted flux tube with helicity $5\Phi^2$ due to twist 3.54 and writhe 1.56 from M Berger (Plas Phys Cont Fus 1999)

Twist and writhe from E Blackman , Space Sci Rev (2014)

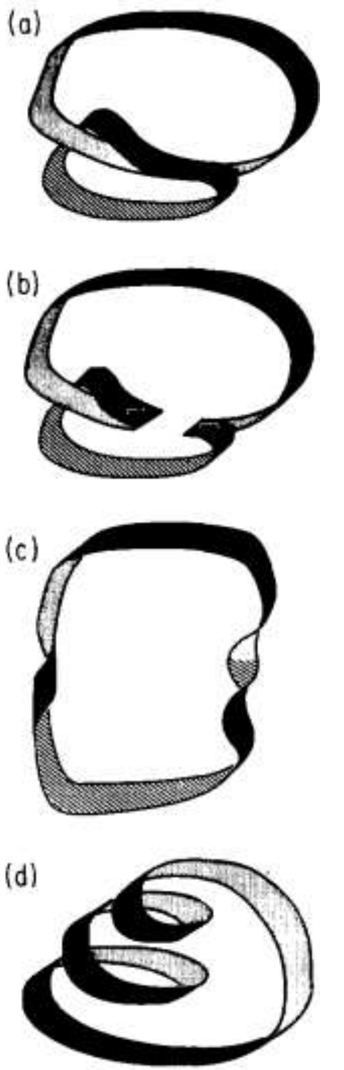


Helicity conservation

- Using the induction equation, it can be shown that helicity is conserved in ideal MHD for every closed volume bounded by a magnetic flux surface

$$\frac{dK}{dt} = 0$$

- Since magnetic topology is preserved in ideal MHD! (See Section 2)
- Taylor hypothesised that in a turbulent plasma, the global helicity is conserved whilst individual flux tube helicities are not
  - Magnetic helicity is approximately conserved in the presence of turbulence/magnetic reconnection whilst energy is dissipated
  - *J B Taylor, Phys Rev Lett 1974*



Helicity conservation during reconnection From Pfister and Gekelman 1991

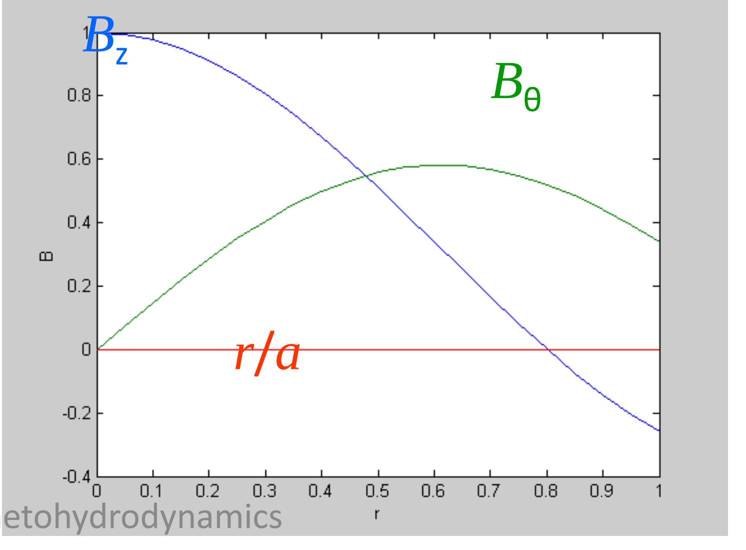
Relaxation theory

- A turbulent plasma will relax towards a state of minimum magnetic energy with conserved helicity
- It can be shown that this relaxed or minimum energy state is a **constant-α force-free field** (see Section 4)

$$\nabla \times \mathbf{B} = \alpha \mathbf{B}, \text{ where } \alpha \text{ is constant} \tag{3.4}$$

- In cylindrical coordinates*, the solution to this is given in terms of Bessel functions as

$$B_z = B_0 J_0(\alpha r)$$
$$B_\theta = B_0 J_1(\alpha r)$$



*See equation (4.7)

4. MAGNETOSTATIC EQUILIBRIUM AND FORCE-FREE FIELDS

Equilibrium

- If the plasma flows are weak and the fields are slowly-evolving, we can set LHS of momentum equation to zero – we have an **equilibrium** in which all forces balance

$$\mathbf{j}\times\mathbf{B}-\nabla p+\rho\mathbf{g}=\mathbf{0} \qquad (4.1)$$

- This can be done if any flows v are weak compared with Alfven speed and sound speed; and time-scales of variation T are slow compared with Alfven time and sound time

$$v \ll v_A, c_s, \quad T \gg L/v_A, L/c_s$$

Alfven speed is the propagation speed of magnetic waves carrying information about changes in $\mathbf{B}$
Alfven waves propagate along magnetic field lines, restoring force is magnetic tension
See Nakariakov lecture

- If we can also neglect gravity we have a **magnetostatic** field

$$\mathbf{j}\times\mathbf{B} = \nabla p \qquad \text{where } \mathbf{j} = \frac{1}{\mu_0}\left(\nabla\times\mathbf{B}\right) \tag{4.2}$$

Gravity can be neglected if length-scale *L* is much less than gravitational scale-height $\Lambda = k_B T/mg$

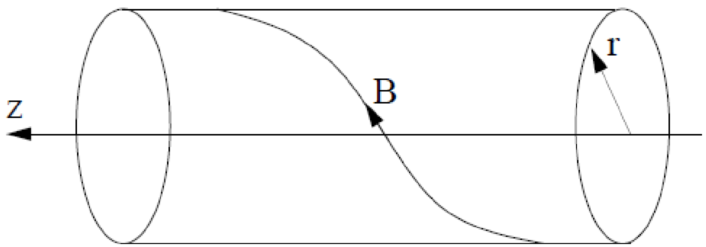
- If the plasma beta is also small $\beta << 1$

then Lorentz force is dominant and we have a **force-free field**

$$\mathbf{j}\times\mathbf{B} = \mathbf{0} \tag{4.3}$$

Magnetostatic fields in a cylinder

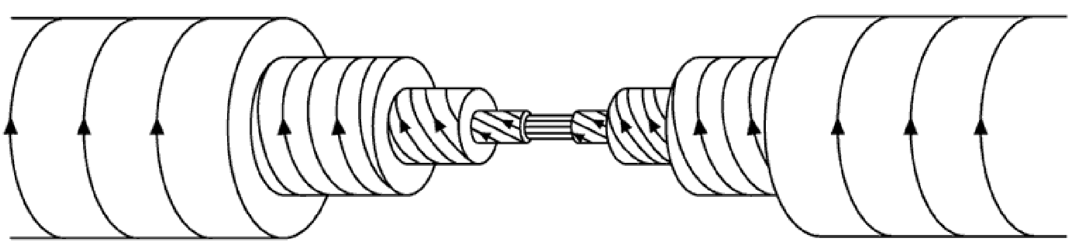
In a cylindrical field where all quantities depend only on *r*, the magnetostatic equation can be written as an ODE



$$\frac{d}{dr}\left(\frac{B_z^2+B_\theta^2}{2\mu_0}+p\right)+\frac{B_\theta^2}{\mu_0 r}=0 \quad (4.4)$$

Total pressure gradient + magnetic tension (circular B_θ lines) = 0

- A "**flux rope**" e.g. sunspot, coronal loop (ignoring curvature), linear laboratory device (e.g. Z pinch), flux rope in planetary atmosphere or solar wind.....



Interior Structure of Flux Rope

Theta-pinch

- Consider a system with **only axial field** B_z (straight field lines) and thermal pressure ($B_\theta = 0$) – then **total pressure is constant**

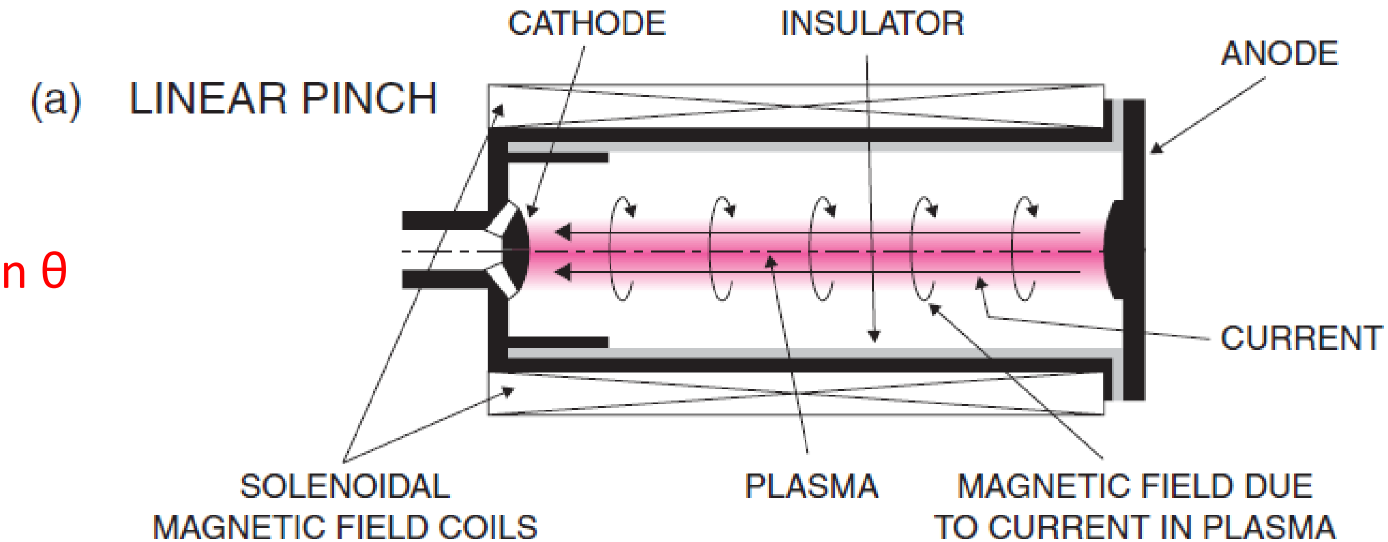
$$\frac{dp_{tot}}{dr} = 0 \Rightarrow p_{tot} \equiv p + \frac{B^2}{2\mu_0} = \text{constant}$$

- In a laboratory, this is a device in which magnetic field is generated by an external solenoid (current in θ direction) – a **θ-pinch**
- In the Sun, a good example is a sunspot

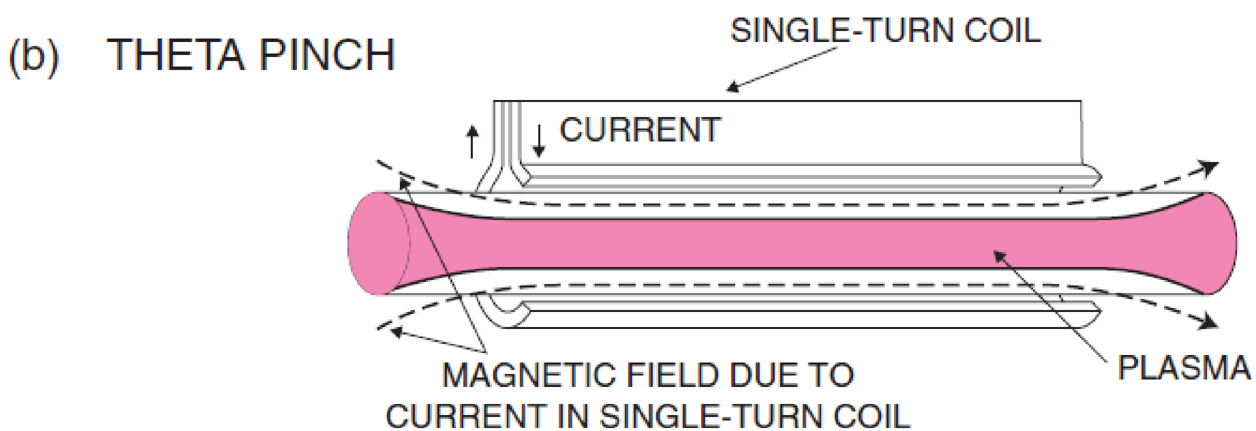
- Note that where *p* is large, *B* is weak – plasma diamagnetism

Z pinch (linear pinch) and Theta pinch

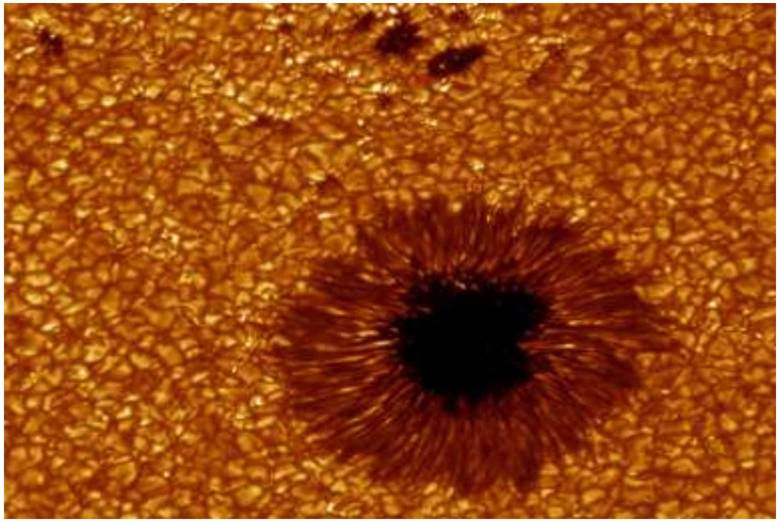
Current in z direction, field in θ direction

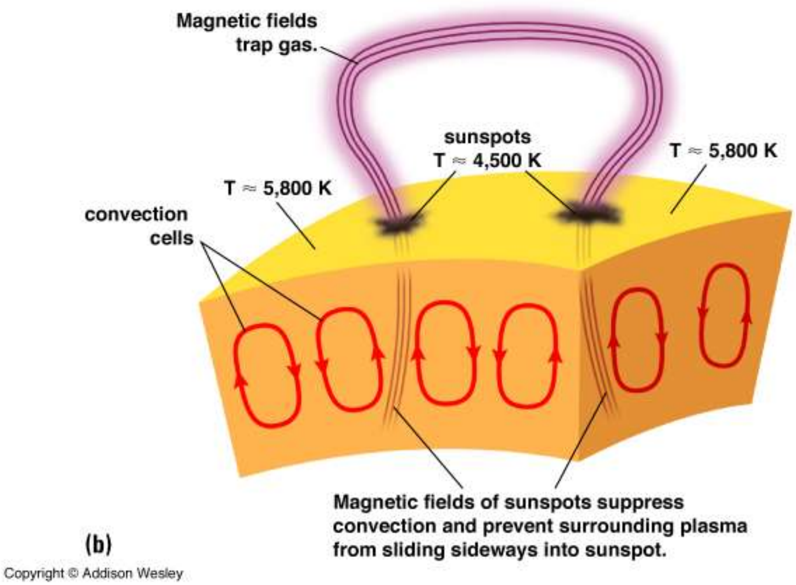


Current in θ direction, field in z direction



Lightning – inward collapse due to current - similar to Z pinch





Sunspots – roughly cylindrical bundle of magnetic field – similar to theta pinch

Z-pinch (cylindrical pinch)

- Consider linear magnetic confinement device with **plasma current in z-direction** generating azimuthal magnetic field B_θ
- Plasma is confined by inwards force or **pinch** - due to tension of circular field lines B_θ – equivalently, axial current *I* provides confinement (mutual attraction of current filaments)

Example Z pinch of radius *a* with uniform current density $j_{0,}$

$$2\pi r B_\theta = \mu_0 \int_0^r 2\pi r\, j_0 dr \Rightarrow B_\theta = \mu_0 j_0 r/2$$

Ampere's Law

$$p(r) = p(0) - \frac{B_\theta^2}{2\mu_0} - \int_0^r \frac{B_\theta^2}{\mu_0} dr$$

Integrating (3.6)

$$\Rightarrow p(r) = p(0) - \mu_0 \frac{j_0^2 r^2}{4}$$

Substituting for B_θ and doing integral

Since pressure at edge vanishes, $p(a) = 0$. Hence central pressure is

$$p(0) = \frac{\mu_0 I^2}{4\pi^2 a^2}$$

In terms of total plasma current $I = \pi a^2 j_0$

Force-free fields

- Consider equilibrium for very low β plasmas (neglect pressure)

$$\mathbf{j} \times \mathbf{B} = \mathbf{0} \Rightarrow (\nabla \times \mathbf{B}) \times \mathbf{B} = \mathbf{0} \qquad (4.5)$$

- Currents flow parallel to magnetic field

$$\nabla \times \mathbf{B} \ // \ \mathbf{B} \Rightarrow \nabla \times \mathbf{B} = \alpha(\mathbf{r})\mathbf{B} \qquad (4.6)$$

where α is a scalar - it can easily be shown α is constant along field lines

- If α is spatially-uniform, this is called a constant-α field or linear force-free field
- For cylindrically-symmetric fields (3.6) becomes

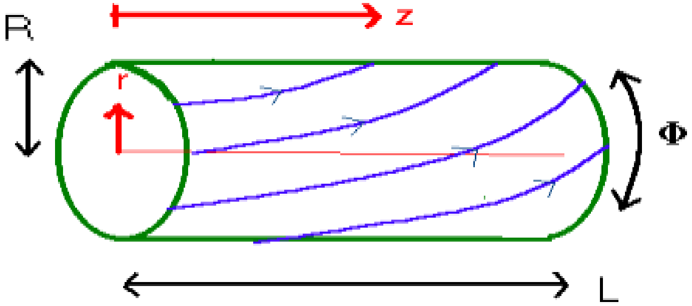
$$\frac{d}{dr}\left(\frac{B_z^2 + B_\theta^2}{2}\right) + \frac{B_\theta^2}{r} = 0 \qquad (4.7)$$

Magnetic pressure gradient

Magnetic tension

Force-free fields in a cylinder
(twisted flux ropes)

- Need one additional condition to fully determine fields



- The angle rotated by a field line Φ in a cylindrical flux tube is given by

$$\Phi(r) = \int_0^L d\theta = \int_0^L \frac{B_\theta}{rB_z} dz = \frac{LB_\theta}{rB_z} \qquad (4.8)$$

Example Constant twist field - assume helical pitch of all field lines is the same

$$\text{Let } \Phi(r) = LB_\theta / rB_z = \Phi_0$$

$$\Rightarrow B_\theta = \Phi_0 \, rB_z / L$$

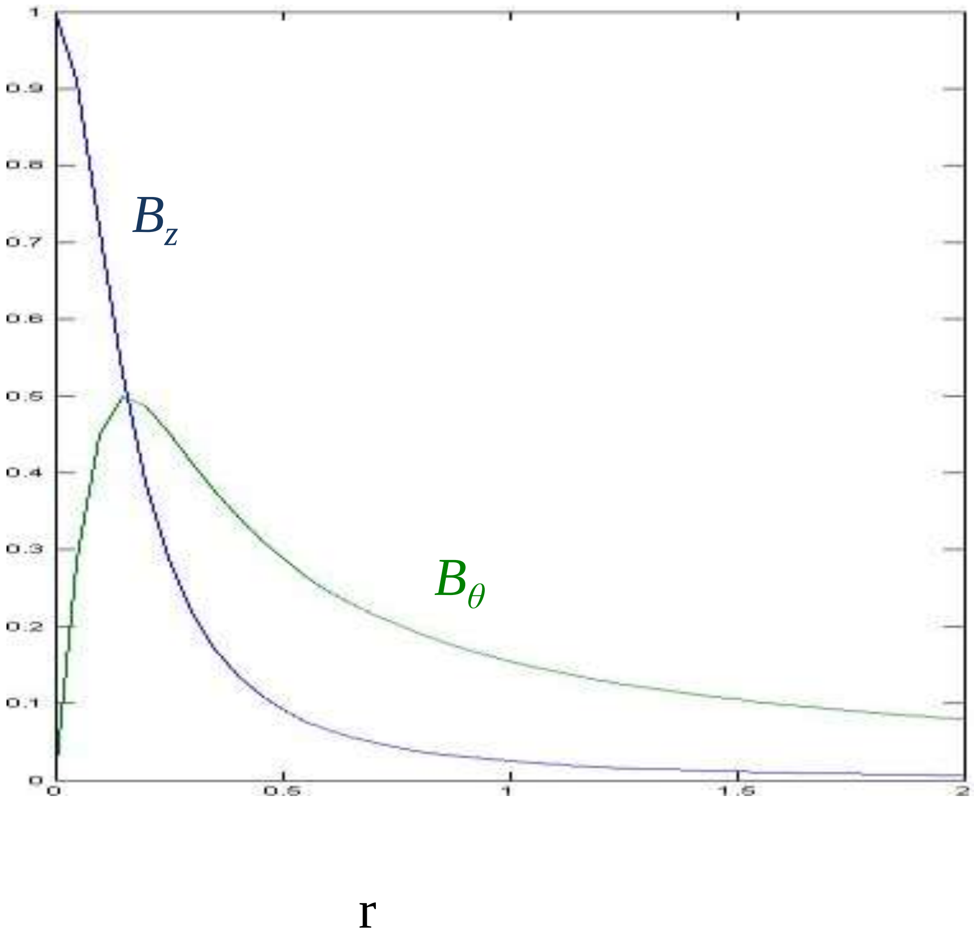
Substitute into (4.7)

$$\Rightarrow \frac{1}{2}\frac{d}{dr}\left[B_z^2 + \frac{r^2\Phi_0^2}{L^2}B_z^2\right] + \frac{1}{r}\frac{r^2\Phi_0^2}{L^2}B_z^2 = 0$$

$$\Rightarrow \frac{d}{dr}\left[B_z^2\left(1+\frac{r^2\Phi_0^2}{L^2}\right)^2\right] = 0$$

Check by working backwards using product rule

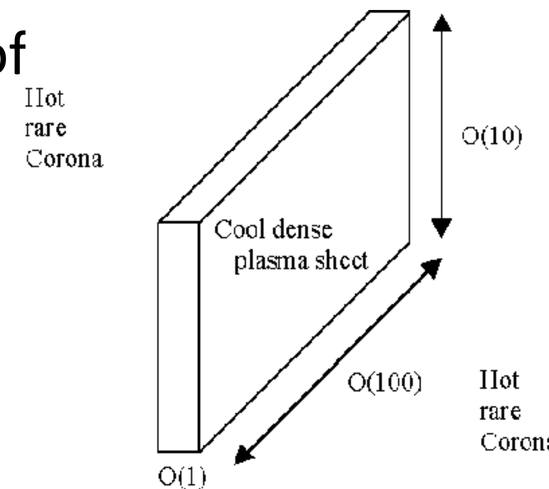
$$\Rightarrow B_z = \frac{B_0}{\left(1+r^2\Phi_0^2/L^2\right)}, \quad B_\theta = \frac{B_0\Phi_0 \, r/L}{\left(1+r^2\Phi_0^2/L^2\right)}$$



$$\frac{\Phi_0}{L} = 2\pi$$

Equilibrium fields involving gravity e.g. Solar prominences

- Often in astrophysical fields gravity can play an important or dominant role in the force balance
- **Solar prominences** are dense cool sheets of in the solar atmosphere

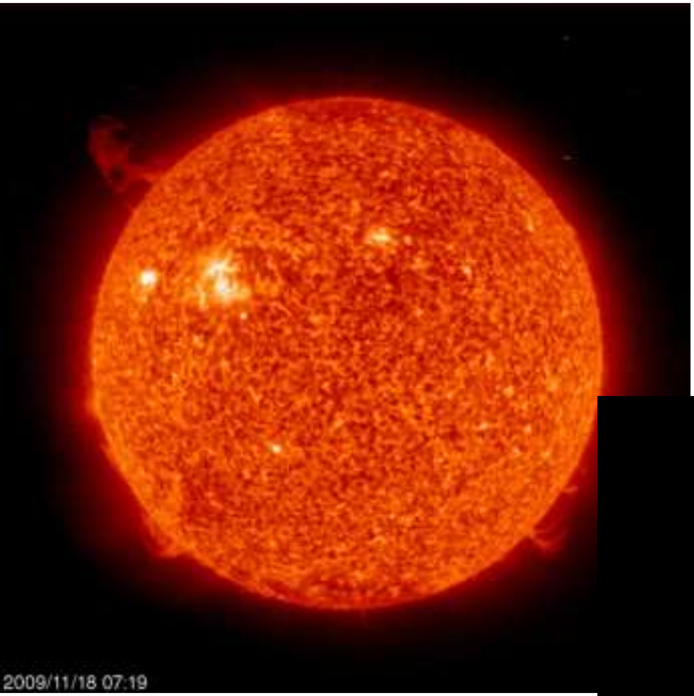


- Observed as dark filaments from above, flame-like sheets from the side
- Typical parameters:

$$n \approx 10^{17}\,\mathrm{m}^{-3},\ T \approx 7000\ \mathrm{K},\ \mathrm{B} \approx 5\text{-}10\times10^{-4}\ \mathrm{T},$$

$$\text{length} \approx 200{,}000\ \text{km, height} \approx 50{,}000\ \text{km, width} \approx 6000\ \text{km}$$

- **Dense plasma supported against gravity mainly by magnetic tension force of curved magnetic field lines**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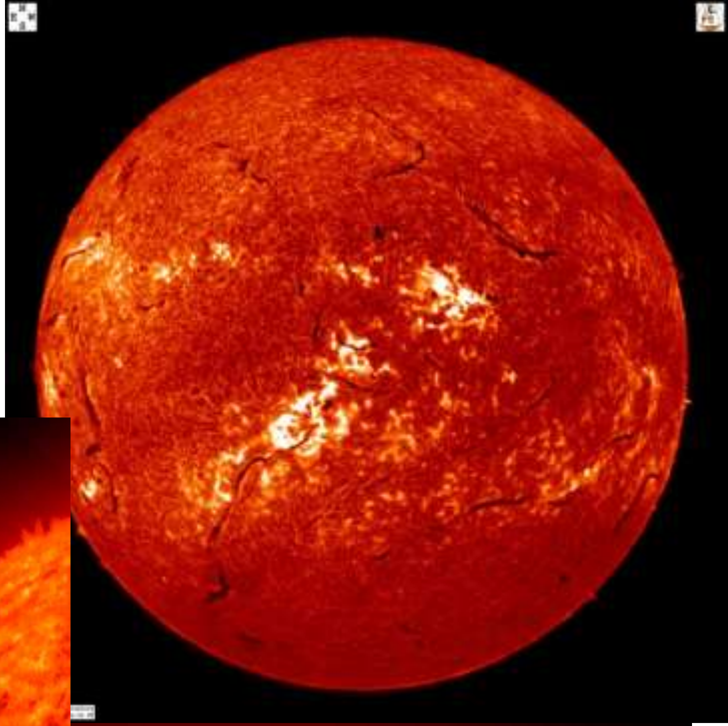
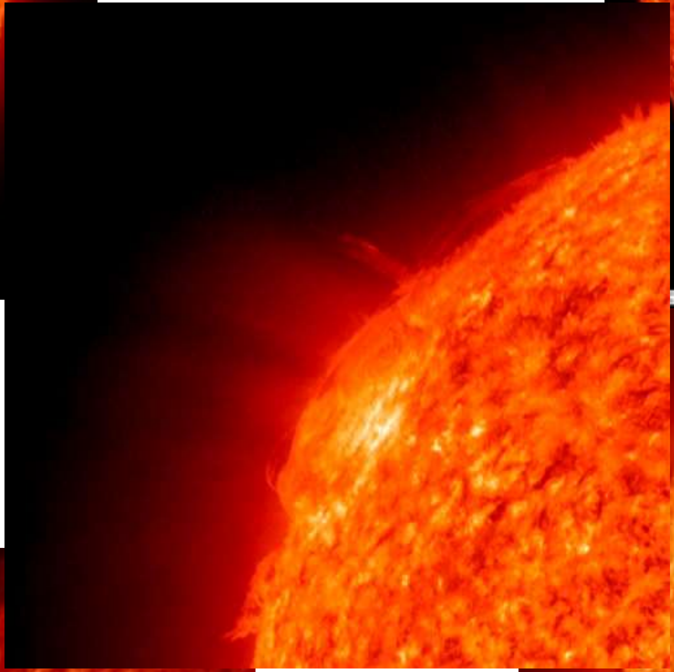
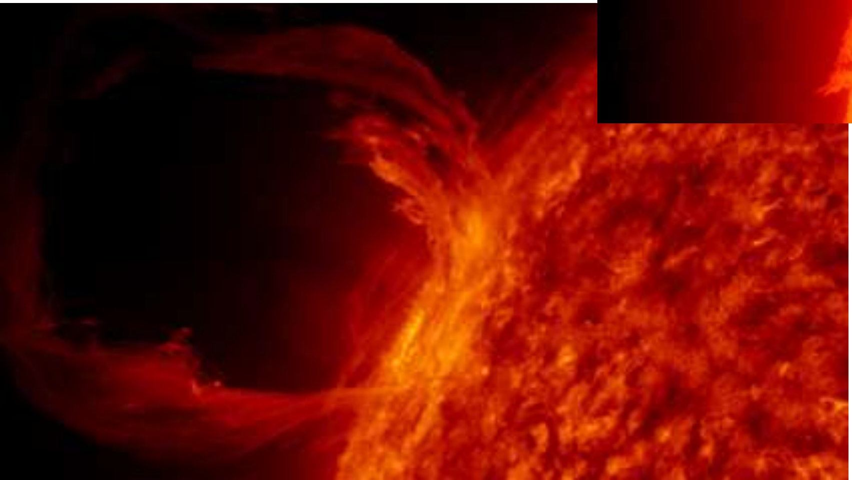
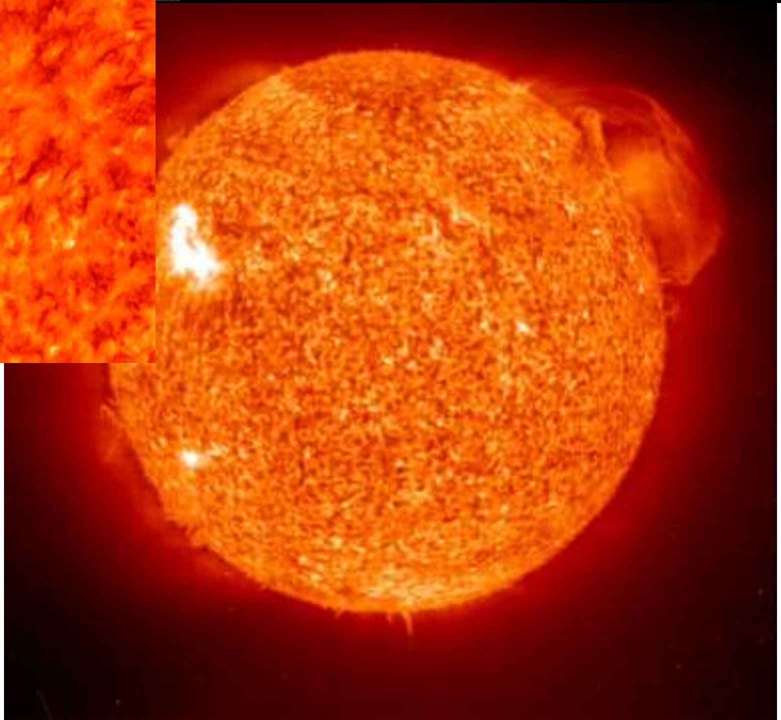








- Simple model of prominence support - **Kippenhahn-Schluter**
- Assume constant T (isothermal), constant horizontal field B_x, B_y; all other quantities depend only on x (coordinate across width)

$$\mathbf{j}\times\mathbf{B} = (1/\mu_0)(0, -dB_z/dx)\times(B_x, B_y, B_z)$$

$$-\frac{dp}{dx} - \frac{1}{\mu_0} B_z \frac{dB_z}{dx} = 0 \Rightarrow p + \frac{B_z^2}{2\mu_0} = \text{const};$$

Horizontal (x) force balance

$$\frac{1}{\mu_0} B_x \frac{dB_z}{dx} = \rho g \Rightarrow \frac{kT}{\mu_0 mg} B_x \frac{dB_z}{dx} - p = 0.$$

Vertical (z) force balance; use gas law

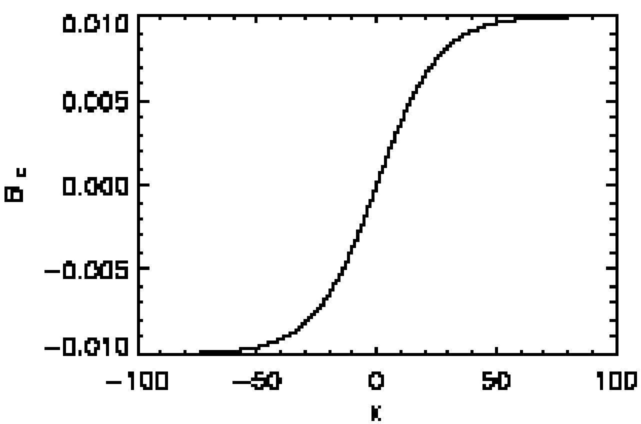
Combining and solving:

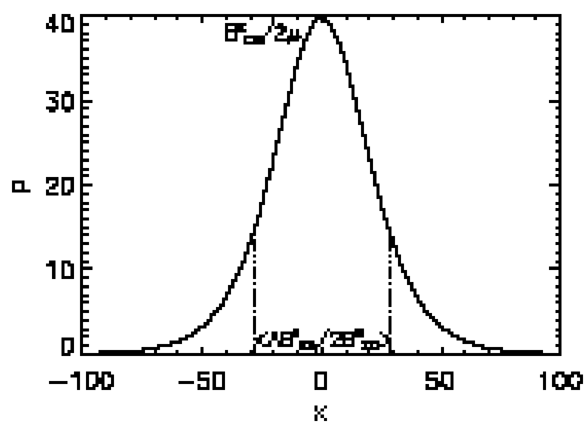
$$\Lambda B_x \frac{dB_z}{dx} + \frac{B_z^2}{2} = \text{const} = \frac{B_0^2}{2} \Rightarrow$$

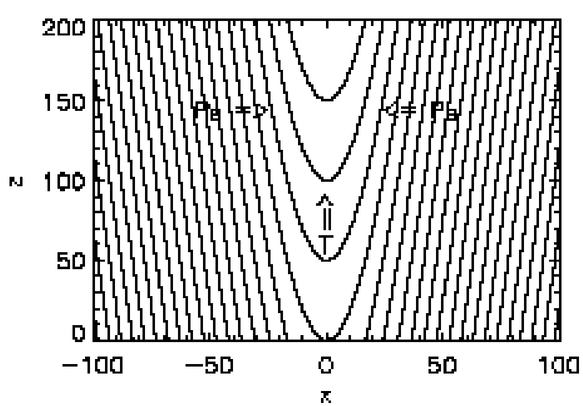
$$B_z = B_0 \tanh\left(\frac{x}{w}\right) \text{ where } w = \frac{2B_x\Lambda}{B_0}.$$

where $\Lambda = kT/mg$ is gravitational scale height

Vertical – balance between magnetic tension of curved field lines (up) and gravity (down)
Horizontal – total pressure constant (plasma pressure peaks at centre of prominence where magnetic pressure dips)





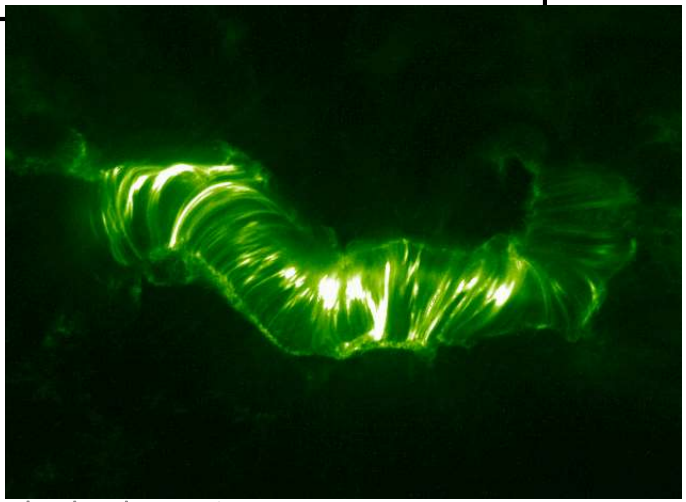


Equilibrium magnetic fields in 2D

- Consider a magnetostatic field in which all quantities are independent of z coordinate
- e.g. Arcade of solar coronal loops
- Express field components in terms of flux function ψ

$$\nabla \cdot \mathbf{B} = 0 \Rightarrow \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} = 0$$

$$\Rightarrow B_x = \frac{\partial \psi(x,y)}{\partial y},\ B_y = -\frac{\partial \psi(x,y)}{\partial x} \quad \text{or } \mathbf{B} = \nabla \psi(x,y) \times \hat{\mathbf{z}} + B_z \hat{\mathbf{z}} \qquad (4.9)$$



Exercises for student – check div**B** = 0 and ψ = constant along field lines
Also show $\psi = A_z$ where **A** is vector potential

- First, take scalar product of force balance (4.2) with **B**

$$(\mathbf{j}\times\mathbf{B}).\mathbf{B} = \mathbf{B}.\nabla p \Rightarrow \mathbf{B}.\nabla p = 0$$

Pressure constant along field lines

$$\Rightarrow \frac{\partial\psi}{\partial y}\frac{\partial p}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial p}{\partial y} = 0$$

Substituting for B_x , B_y

Jacobian vanishes so p is a function of ψ – called P

$$\Rightarrow p = P(\psi)$$

- Pressure is constant along field lines
- Now calculate current (from Ampere's Law)

$$\mathbf{j} = \frac{1}{\mu_0}\nabla\times\mathbf{B} = \frac{\partial B_z}{\partial y}\hat{\mathbf{x}} - \frac{\partial B_z}{\partial x}\hat{\mathbf{y}} + \left(-\frac{\partial^2\psi}{\partial x^2} - \frac{\partial^2\psi}{\partial y^2}\right)\hat{\mathbf{z}}$$

$\nabla^2\psi$

- Substitute into force-balance equation (4.2):

z-component:

Invariance in z

$$(\mathbf{j}\times\mathbf{B})_z = \frac{\partial p}{\partial z} = 0$$

$$\Rightarrow \frac{\partial B_z}{\partial y}\frac{\partial \psi}{\partial x} - \frac{\partial B_z}{\partial x}\frac{\partial \psi}{\partial y} = 0$$

$$\Rightarrow B_z = B_z(\psi)$$

Substituting for B_x , B_y

Jacobian vanishes so B_z is a function of ψ

x-component:

$$(\mathbf{j}\times\mathbf{B})_x = \frac{\partial p}{\partial x} = \frac{dP}{d\psi}\frac{\partial \psi}{\partial x}$$

Substituting $B_z = B_z(\psi)$

$$\Rightarrow -\frac{1}{\mu_0}\frac{\partial \psi}{\partial x}\frac{dB_z}{d\psi}B_z - \nabla^2\psi\frac{\partial \psi}{\partial x} = \frac{dP}{d\psi}\frac{\partial \psi}{\partial x}$$

Grad-Shafranov equation

•Finally we obtain the Grad-Shafranov equation (also from y component):

$$\nabla^2\psi + B_z \frac{dB_z}{d\psi} + \mu_0 \frac{dP}{d\psi},$$

where axial field $B_z = B_z(\psi)$

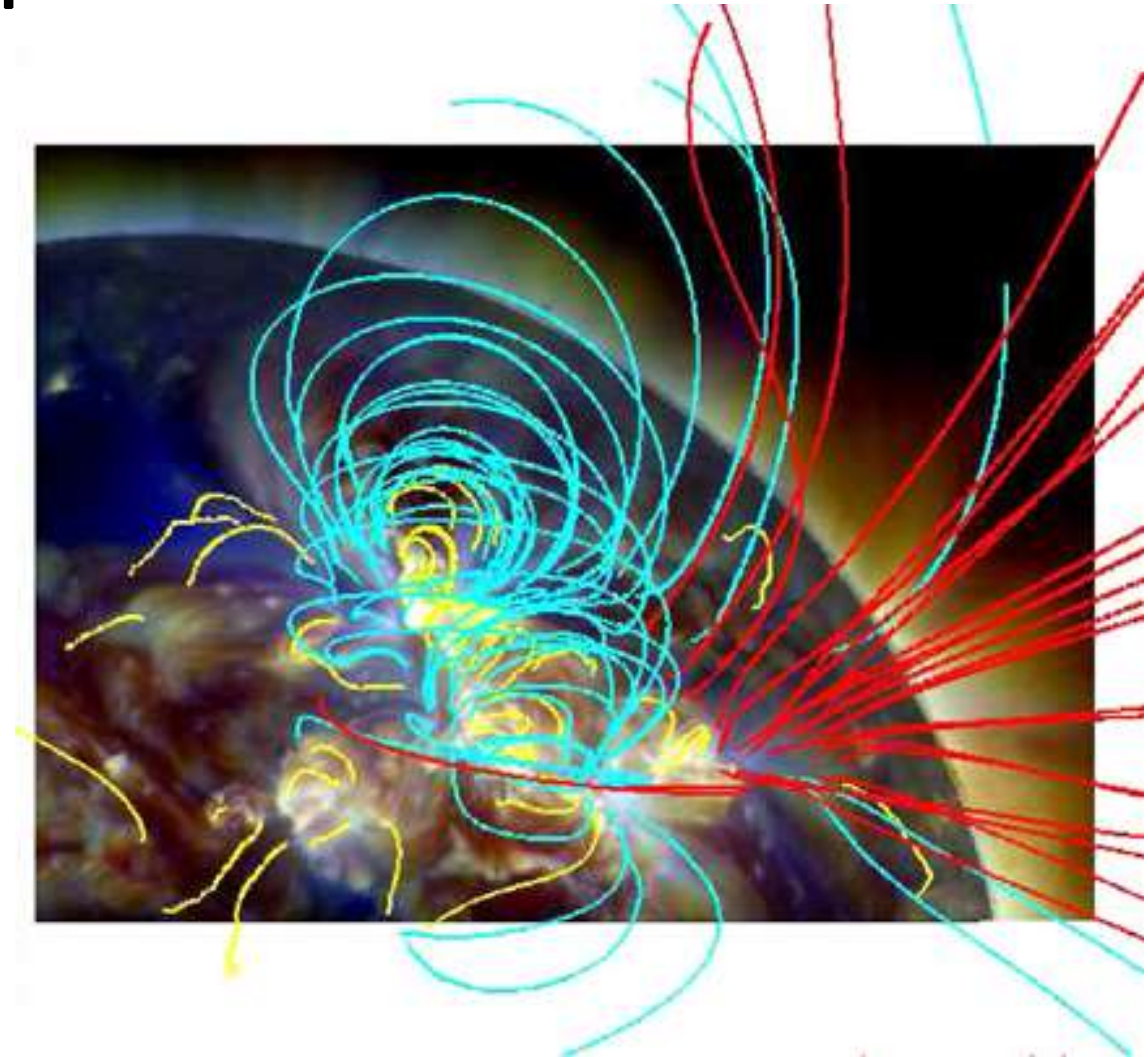
and pressure $p = P(\psi)$

•A second-order nonlinear elliptic partial differential equation
•In general, must solve numerically

•Useful special case – constant-α field - $B_z = \alpha\psi,\ P = 0$

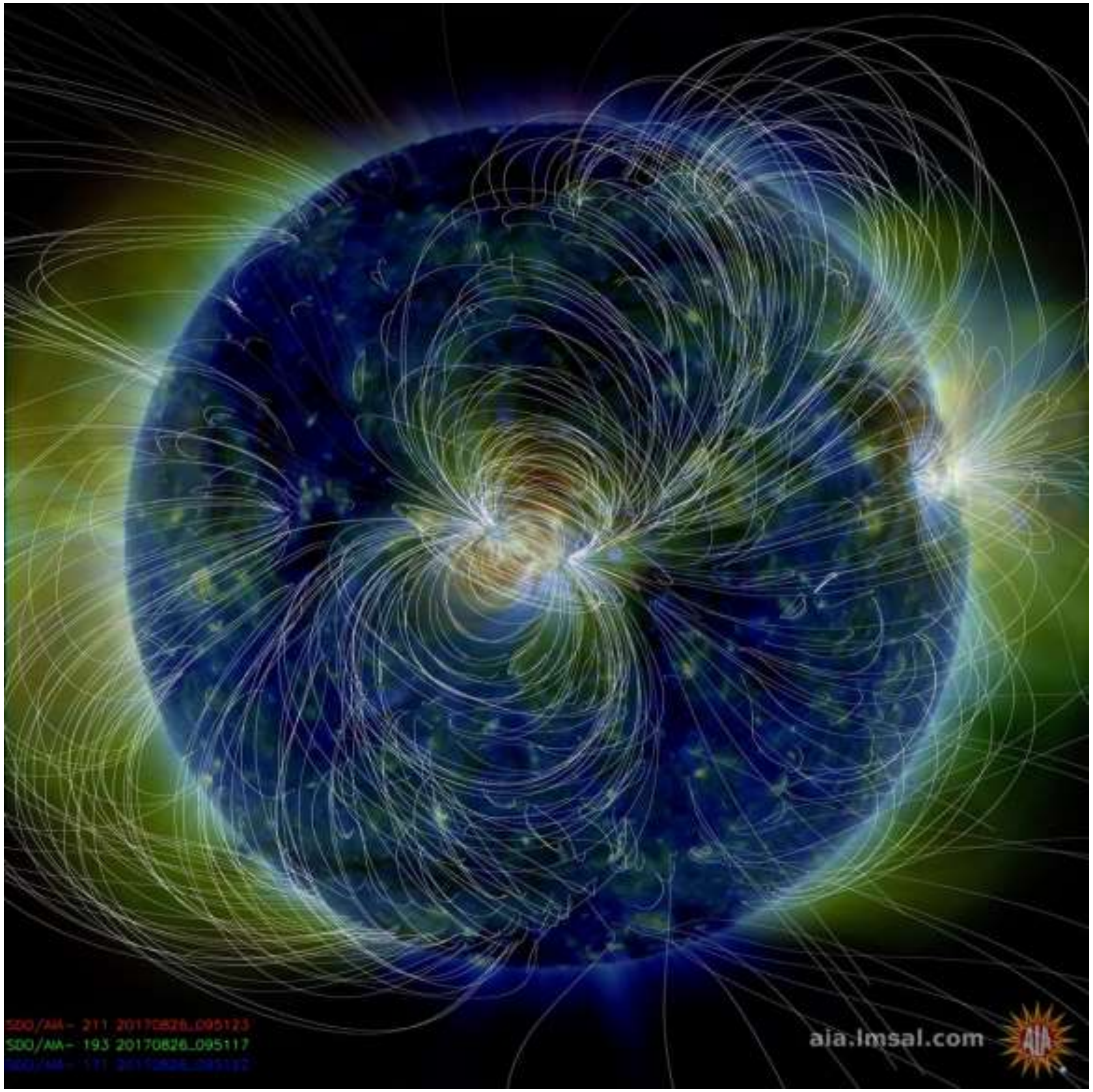
•Linear, may solve by separation of variables etc

3D equilibria – force-free field in solar corona



From Guo et al, Ap J 2012

Global coronal magnetic field model



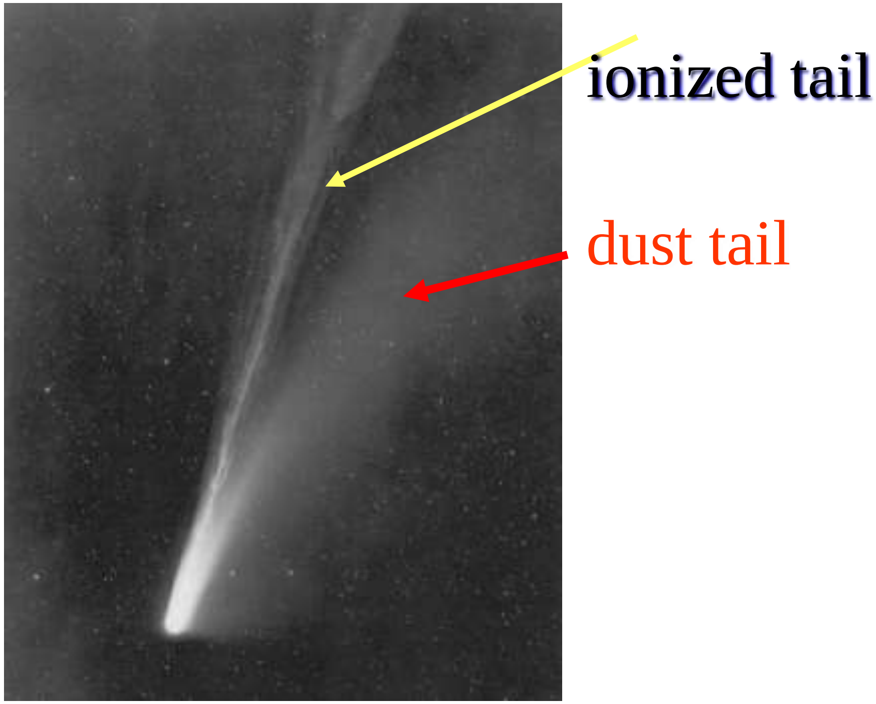
- **PFSS**
- Potential Field Source Surface
- Simplest case of force-free field **j = 0**
- Matches to radial magnetic field of solar wind at source surface

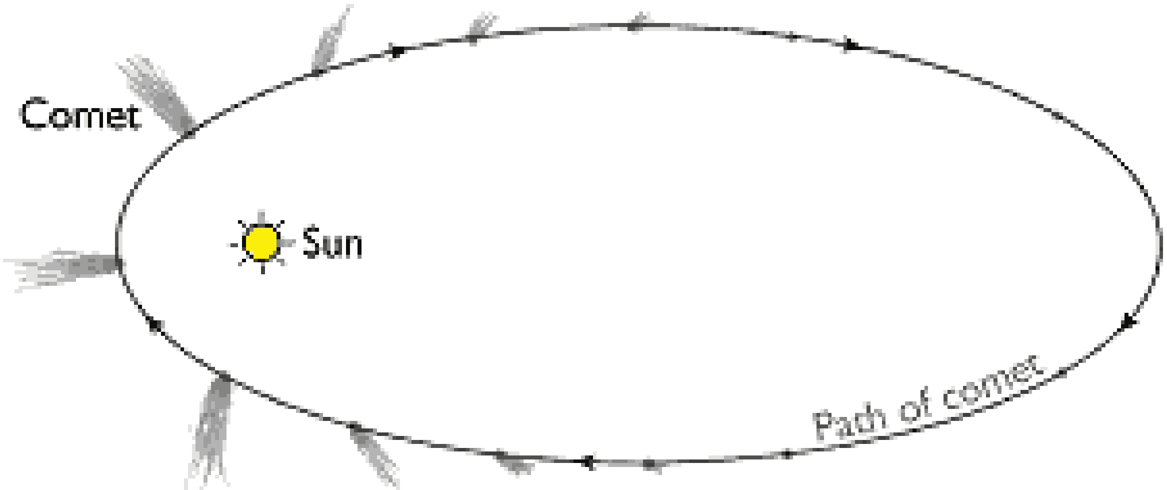
5. Flowing plasma – the solar wind

Flows – the solar wind

- Some early evidence for existence of a solar wind was given by comet tails – these always point away from Sun
- Solar wind then predicted theoretically by Parker (1958) and confirmed by direct evidence of moving plasma by space craft from 1960s onwards
  - First direct in situ measurements of flowing plasma from Mariner
  - Voyager has measured solar wind velocity directly – to the edge of the heliosphere
  - Solar Wind velocity routinely measured by spacecraft near Earth e.g. ACE, WIND
  - ULYSSES measured solar wind out of the ecliptic plane (over poles of Sun)

Observations of comet tails suggested outflow from Sun





The solar wind equation

- Consider steady-state, spherically-symmetric, isothermal corona (temperature *T*) with radial flow *u.* Assume fully-ionised hydrogen plasma which is a perfect gas; neglect magnetic force.
- Momentum equation:

$$\rho u \frac{du}{dr} = -\frac{dp}{dr} - \frac{GM\rho}{r^2} \tag{5.1}$$

- Mass conservation equation:

$$\frac{d}{dr}\left(4\pi r^2 \rho u\right) = 0 \tag{5.2}$$

(constant flux of plasma through every spherical surface)

- Perfect gas law:

$$p = \frac{k}{m_p/2}\rho T \tag{5.3}$$

- Combine (eliminating p and ρ),derive **the Solar Wind equation**

$$\left(u^2 - \frac{2kT}{m_p}\right)\frac{1}{u}\frac{du}{dr} = \frac{4kT}{m_p r} - \frac{GM_o}{r^2} \qquad (5.4)$$

- This is a first order Ordinary Differential Equation which can easily be integrated to give

$$\frac{1}{2}u^2 - \frac{2kT}{m_p}\ln u = \frac{4kT}{m_p}\ln r + \frac{GM_o}{r} + C$$

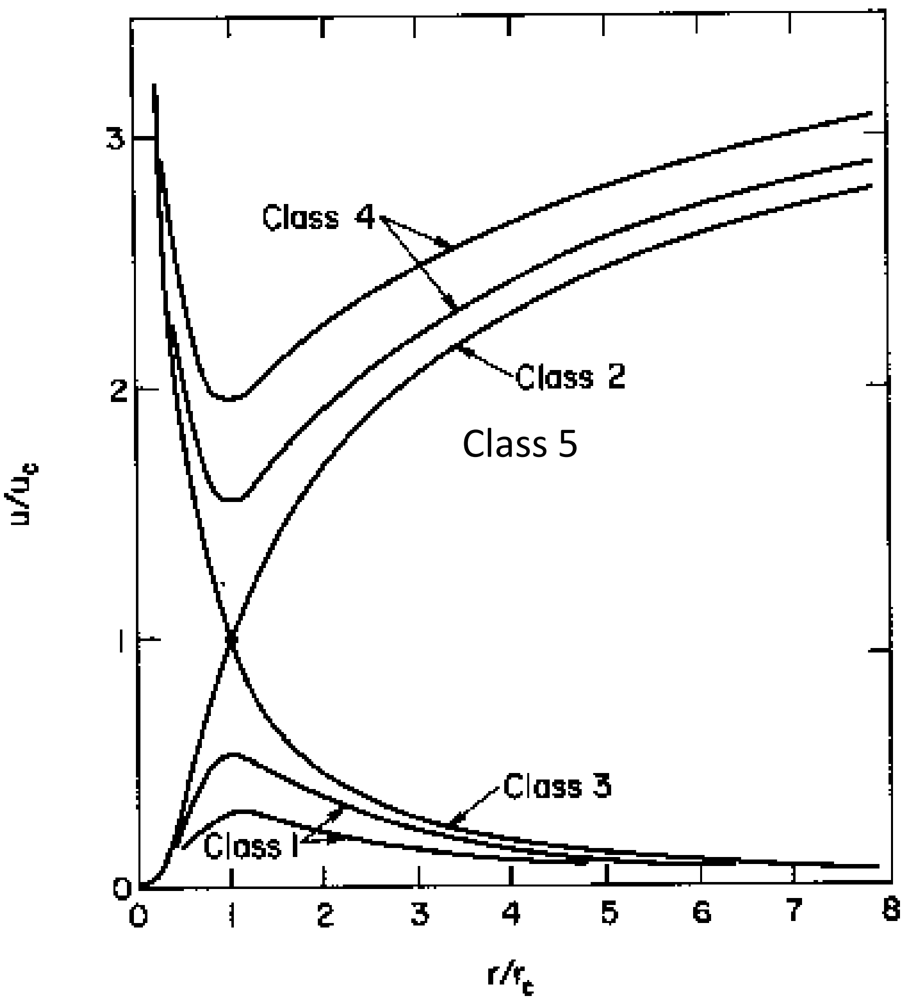
where C is a constant of integration (hence family of solutions)

- The RHS vanishes at the **critical radius:**

$$r_c = \frac{\mathrm{GM_o} m_p}{4kT}$$

- At critical radius, either *du/dr =* 0 (*u(r)* has a turning point) or $u = c_s$ (flow speed is sonic)

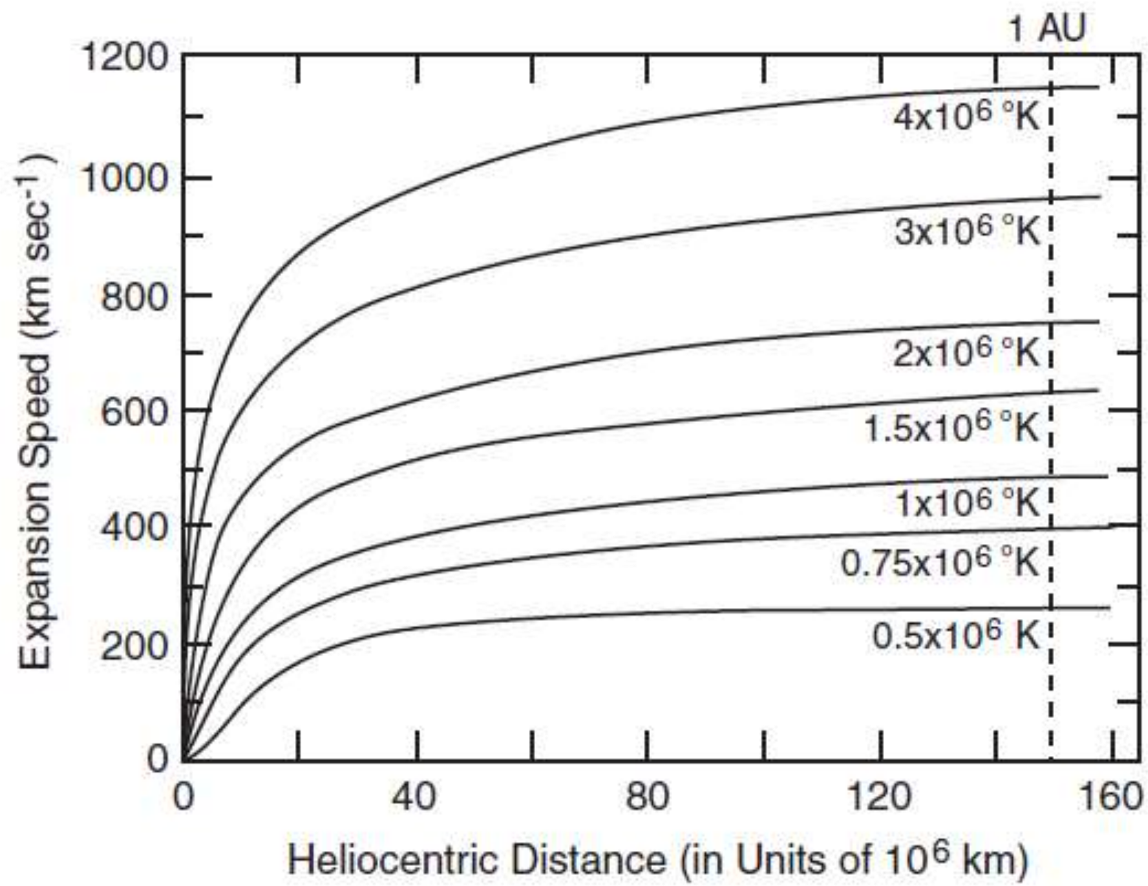
Selection of solar wind solution



- Solutions in classes 1, 3, 4, 5 can be ruled out – only relevant solution is class 2
- Solar wind solution starts with low speed at solar surface – becomes supersonic beyond r_c

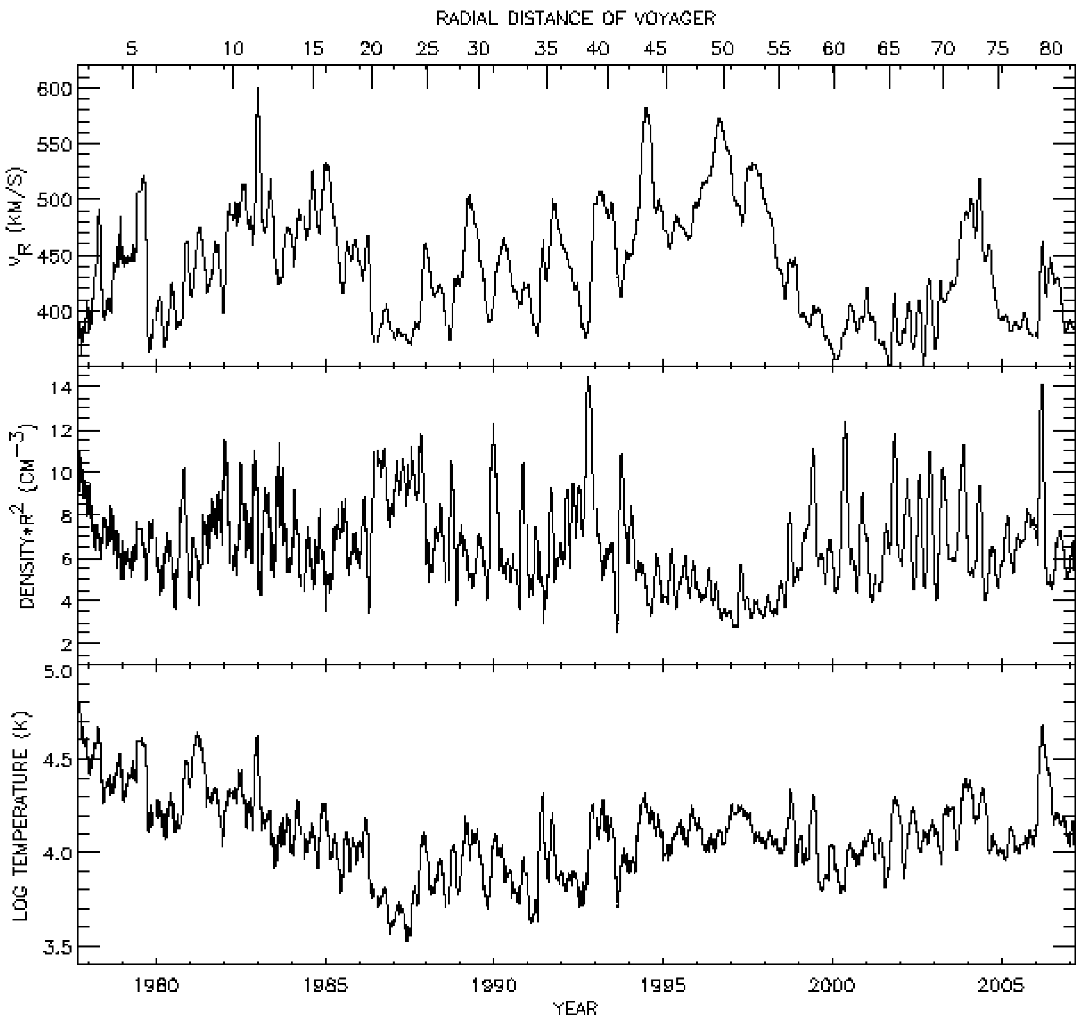
Parker solar wind solution

Dependence of u on r (for different values of temperature T)



Speed u is increasing function of distance r

Voyager 2 – solar wind overview



Summary

- Magnetohydrodynamics (MHD) provides a powerful tool for modelling the interaction of a plasma – treated as an electrically-conducting fluid – with magnetic fields
- May be used to model
  - Solar interior
  - Solar atmosphere
  - Global properties of Solar Wind and planetary magnetospheres
  - And more (e.g. laboratory plasmas)
- In the following lectures, you will learn about:
  - MHD waves and instabilities
  - Magnetic reconnection
  - And many other topics involving MHD models

Reading list

- “Magnetohydrodynamics of the Sun” E R Priest
- “Principles of magnetohydrodynamics” H Goedbloed and S Poedts
- “Lectures in magnetohydrodynamics” D D Schnack
- “Ideal magnetohydrodynamics” J P Freidberg